3.2 The Growth of Functions

Big-O Notation

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say $f(x)$ is $O(g(x))$ if there are constants C and k such that

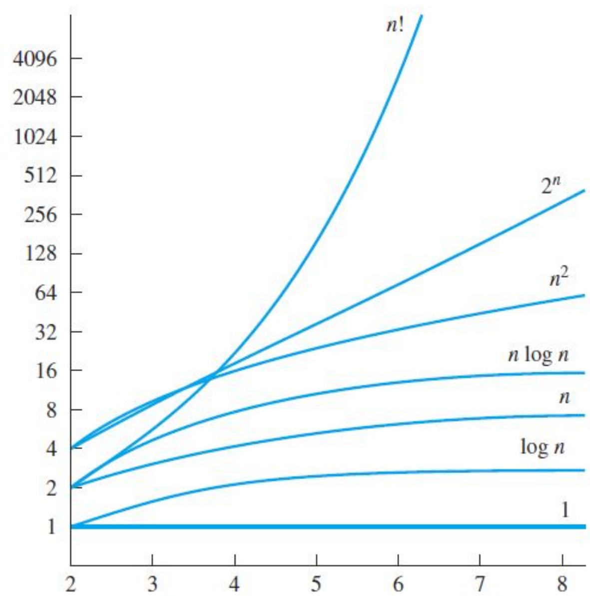
$$|f(x)| \leq C|g(x)|$$

whenever $x > k$.
In other words, Big-O is the upper bound for the growth of a function.

Important Complexity Classes

These are common functions for big-O from least to greatest:

$$1, \log n, n, n \log n, n^2, 2^n, n!$$



The Growth of Combinations of Functions

If $f_1(x) = O(g_1(x))$ and $f_2(x) = O(g_2(x))$, then

- $(f_1 + f_2)(x) = O(\max(|g_1(x)|, |g_2(x)|))$
- $(f_1 f_2)(x) = O(g_1(x) g_2(x))$

Big-Ω Notation

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say $f(x)$ is $\Omega(g(x))$ if there are positive constants C and k such that

$$|f(x)| \geq C|g(x)|$$

whenever $x > k$.

Big-Θ Notation

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(x)$ is $\Theta(g(x))$ if $f(x)$ is $O(g(x))$ and $f(x)$ is $\Omega(g(x))$. Note that $f(x)$ is $\Theta(g(x))$ if and only if there are positive constants C_1, C_2, and k such that

$$C_1|g(x)| \leq f(x) \leq C_2|g(x)|$$

whenever $x > k$.

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Determine whether each of these functions is $O(x)$.

a) $f(x) = 10$

Yes. $|10| \leq |x|$ for all $x > 10$ with our witnesses $C = 1$ and $k = 10$.

b) $f(x) = 3x + 7$

Yes. $|3x + 7| \leq 4|x|$ for all $x > 7$ with our witnesses $C = 4$ and $k = 7$.

c) $f(x) = x^2 + x + 1$

No. There is no value C and no value k where $|x^2 + x + 1| \leq C|x|$ for large values of x.

d) $f(x) = 5 \log x$

Yes. $|5 \log x| \leq 5|x|$ for all $x > 1$ with our witnesses $C = 5$ and $k = 1$.

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Show that $(x^2 + 1)/(x + 1)$ is $O(x)$

Simplify fraction first.
$\dfrac{x^2+1}{x+1} = \dfrac{x^2-1+2}{x+1}$
$= \dfrac{x^2-1}{x+1} + \dfrac{2}{x+1}$
$= \dfrac{(x+1)(x-1)}{x+1} + \dfrac{2}{x+1}$
$= x - 1 + \dfrac{2}{x+1}$
Now that the fraction has been simplified, we can find the big-O.
$x - 1 + \dfrac{2}{x+1} \leq x$ for $x > 1$

The function is $O(x)$ with our witnesses $C = 1$ and $k = 1$.

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Find the least integer n such that $f(x)$ is $O(x^n)$ for each of these functions.

a) $f(x) = 2x^3 + x^2 \log x$

$2x^3 + x^2 \log x \leq 2x^3 + x^3$ for $x > 0$
$2x^3 + x^2 \log x \leq 3x^3$
$O(x^3)$ with our witnesses $C = 3$ and $k = 0$.
Therefore, $n = 3$.

b) $f(x) = 3x^3 + (\log x)^4$

$3x^3 + (\log x)^4 \leq 3x^3 + x^3$ for $x > 1$
$3x^3 + (\log x)^4 \leq 4x^3$
$O(x^3)$ with our witnesses $C = 4$ and $k = 1$
Therefore, $n = 3$.

c) $f(x) = (x^4 + x^2 + 1)/(x^3 + 1)$

Simplify fraction first.
$\dfrac{x^4 + x^2 + 1}{x^3 + 1} = x + \dfrac{1}{x+1}$.
Now that the fraction has been simplified, we can find the big-O.
$x + \dfrac{1}{x+1} \leq x + x$ for $x > 1$
$x + \dfrac{1}{x+1} \leq 2x$
$O(x)$ with our witnesses $C = 2$ and $k = 1$.
Therefore, $n = 1$.

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Give as good a big-O estimate as possible for each of these functions

a) $(n^2 + 8)(n + 1)$

$= n^3 + n^2 + 8n + 8$
Biggest term is n^3 so the function is $O(n^3)$.

b) $(n \log n + n^2)(n^3 + 2)$

$= n^4 \log n + n^5 + 2n \log n + 2n^2$
Biggest term is n^5 so the function is $O(n^5)$.

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Arrange the functions $\sqrt{n}, 1000 \log n, n \log n, 2n!, 2^n, 3^n$, and $n^2/1000000$ in a list so that each function is big-O of the next function.

$1000 \log n, \sqrt{n}, n \log n, n^2/1000000, 2^n, 3^n, 2n!$

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Determine whether each of these functions is $\Omega(x^2)$

c) $f(x) = x \log x$

No. $x \log x$ grows more slowly than x^2, since $\log x$ grows more slowly than x. Therefore $f(x)$ is not $\Omega(x^2)$.

e) $f(x) = 2^x$

Yes. $|2^x| \geq |x^2|$ where $x > 4$ with our witnesses $C = 1$ and $k = 4$.