## 2.3 Functions

Let $A$ and $B$ be nonempty sets. A function $f$ from $A$ to $B$ is an assignment of exactly one element of $B$ to each element of $A$. If $f$ is a function from $A$ to $B$, wee write $f : A \to B$.

#### Domain, Codomain, Image, Preimage, Range

A function from $A$ to $B$:
$f : A \to B$
$A$ is the *domain*
$B$ is the *codomain*

$a \in A$, $b \in B$ such that $f(a) = b$
$a$ is the *preimage* of $b$ under $f$
$b$ is the *image* of $a$ under $f$

The *range* is a specific subset of the Codomain($B$) containing the actual values the function outputs.
The *range* can be written as $f(A)$.

#### Injection (One-to-One)

A function where each element in the Domain maps to a single, unique element in the Codomain. [Domain and Range have the same cardinality]. Strictly increasing or strictly decreasing functions are one-to-one.

#### Surjection (Onto)

A function where every element in the Codomain is a valid output of the function. [Range is equal to Codomain].

#### Bijection

A function that is both an injection and a surjection.

#### Identity Function

A function that maps $f : A \to A$, such that $f(a) = a$ where $a \in A$.

#### Inverse Function

Given the bijective function $f$, such that $f : A \to B$ and $f(a) = b$ where $a \in A$ and $b \in B$, the inverse function is defined as $f^{-1}$, such that $f^{-1} : B \to A$ and $f^{-1}(b) = a$.

**Composition**

Given two functions, $f$ and $g$, such that the range of $g$ is a subset of the domain of $f$, the *composition* of $f$ with $g$ ($f \circ g$) is defined as $f(g(x))$, with $x \in$($g$'s domain).

**Floor Function**

$\lfloor x \rfloor$ returns the largest integer $\leq x$.

**Ceiling Function**

$\lceil x \rceil$ returns the smallest integer $\geq x$.

**2.3 pg 153 # 13**

Determine whether each of these functions from $\mathbb{Z}$ to $\mathbb{Z}$ is onto (surjective).

a) $f(n) = n - 1$

This is surjective since every integer is 1 less than some integer.

b) $f(n) = n^2 + 1$

Not surjective because the range cannot include negative integers.

c) $f(n) = n^3$

Not surjective because any element in the codomain that is not a perfect cube will not be mapped to.

**2.3 pg 153 # 23**

Determine the type of each function from $\mathbb{R}$ to $\mathbb{R}$

a) $f(x) = 2x + 1$

Bijective. This is injective because for every $a \neq b$, we have $f(a) \neq f(b)$ (every number is 1 more than 2 times some number). We also know that the function is surjective because the range is all real numbers from $2((y-1)/2) + 1 = y$.

b) $f(x) = x^2 + 1$

Not injective and not surjective. We know the function is not injective because we can have the same value for $f(x)$ given two different $x$ values. For example, $f(2) = 2^2 + 1 = 5$ and $f(-2) = (-2)^2 + 1 = 5$. The function is also not surjective because the range is all real numbers greater than or equal to 1, or can be written as $[1, \infty)$.

c) $f(x) = x^3$

Bijective. This is injective because for every $a \neq b$, we have $f(a) \neq f(b)$ (every number is the cube of some number). We also know that the function is surjectve because the range is all real numbers from $(y^{1/3})^3 = y$.

d) $f(x) = (x^2 + 1)/(x^2 + 2)$

Not injective and not surjective. We know the function is not injective because we can have the same value for $f(x)$ given two different $x$ values. The function is also not surjective because the range is only $[0.5, 1)$.

**Extra Problem**

Given the following functions $f$ and $g$, from $\mathbb{R}$ to $\mathbb{R}$, find $f \circ g$.

a) $f(x) = x^2$
$g(x) = x + 1$
$(f(g(x)) = f(x + 1) = (x + 1)^2$

b) $f(x) = 2x + 1$
$g(x) = x^2 + 4x + 4$
$(f(g(x)) = f(x^2 + 4x + 4) = 2(x^2 + 4x + 4) + 1 = 2x^2 + 8x + 9$

c) $f(x) = \{(1, 3), (2, 4), (5, 6), (4, 8)\}$
$g(x) = \{(1, 1), (4, 5), (6, 2)\}$
$(f \circ g) = \{(1, 3), (4, 6), (6, 4)\}$

**2.3 pg 154 # 31**

Let $f(x) = \lfloor x^2/3 \rfloor$. Find $f(S)$ if

c) $S = \{1, 5, 7, 11\}$
$f(1) = \lfloor 1^2/3 \rfloor = \lfloor 1/3 \rfloor = 0$
$f(5) = \lfloor 5^2/3 \rfloor = \lfloor 25/3 \rfloor = 8$
$f(7) = \lfloor 7^2/3 \rfloor = \lfloor 49/3 \rfloor = 16$
$f(11) = \lfloor 11^2/3 \rfloor = \lfloor 121/3 \rfloor = 40$
Therefore, $f(S) = \{0, 8, 16, 40\}$

d) $S = \{2, 6, 10, 14\}$
$f(2) = \lfloor 2^2/3 \rfloor = \lfloor 4/3 \rfloor = 1$
$f(6) = \lfloor 6^2/3 \rfloor = \lfloor 36/3 \rfloor = 12$
$f(10) = \lfloor 10^2/3 \rfloor = \lfloor 100/3 \rfloor = 33$
$f(14) = \lfloor 14^2/3 \rfloor = \lfloor 196/3 \rfloor = 65$
Therefore, $f(S) = \{1, 12, 33, 65\}$

**2.3 pg 154 # 43**

Let $g(x) = \lfloor x \rfloor$. Find

a) $g^{-1}(\{0\})$

We need to find the set of all numbers whose floor is 0. Since all number from 0 to 1 (including 0 and excluding 1) round down to 0, then $g^{-1}(\{0\}) = \{x \mid 0 \leq x < 1\}$

b) $g^{-1}(\{-1, 0, 1\})$

We know that the numbers from -1 to 2 (exclusive) round down to either -1, 0, or 1, then $g^{-1}(\{-1, 0, 1\}) = \{x \mid -1 \leq x < 2\}$

c) $g^{-1}(\{x \mid 0 < x < 1\})$

Since $g(x) = \lfloor x \rfloor$ will always result in an integer, no value of $x$ will result in a number between 0 and 1. Thus, the image of the inverse function is the empty set, $\emptyset$

**2.3 pg 155 # 69**

Find the inverse function of $f(x) = x^3 + 1$.

Solve for $x$.
$y = x^3 + 1$
$y - 1 = x^3$
$(y - 1)^{1/3} = x$

The inverse function function is $f^{-1}(x) = (x - 1)^{1/3}$.

**Extra Problem**

For each function from $\mathbb{R}$ to $\mathbb{R}$, if the function has a defined inverse, find it.

a) $f(x) = x^2 - 2$

This function is not bijective, so there is no inverse function.

b) $f(x) = 3$

This function is not bijective, so there is no inverse function.