1.7 Introduction to Proofs

A *proof* is a valid argument that establishes the truth of a statement.

Types of Proofs

- Trivial proof: Prove q by itself
- Vacuous proof: Prove $\neg p$ by itself
- Direct proof: Assume p is true, and prove q
- Proof by Contraposition: Assume $\neg q$, and prove $\neg p$ (Try to prove $\neg q \to \neg p$)
- Proof by Contradiction: Assume $p \wedge \neg q$ and show this leads to a contradiction $((p \wedge \neg q) \to \mathbf{F})$

1.7 pg 91 # 11

Prove or disprove that the product of two irrational numbers is irrational.

To disprove this this proposition, we will find a counterexample. We know that $\sqrt{2}$ is irrational. So by taking the product of $\sqrt{2}$ and $\sqrt{2}$, we obtain 2. 2 is a rational number from the product of two irrational numbers, thus we have disproven the statement.

1.7 pg 91 # 1

Use a direct proof to show that the sum of two odd integers is even.

Assume that the integers a and b are odd. Then there exists two integers n and m such that $a = 2n + 1$ and $b = 2m + 1$. Adding a and b results in $a + b = (2n + 1) + (2m + 1) = 2n + 2m + 2 = 2(n + m + 1)$. Since $a + b$ is equal to two times some integer, we know that the sum of a and b is even by definition of an even integer.

1.7 pg 91 # 13

Prove that if x is irrational, then $1/x$ is irrational.

We will use proof by contraposition. The contrapositive is "If $1/x$ is rational, then x is rational." Suppose that $1/x$ is rational and $x \neq 0$. Then there exists integers p and q such that $1/x = p/q$ and $q \neq 0$. $\dfrac{1}{x} \neq 0$ because $1 \neq x \cdot 0$, this would mean that $p \neq 0$. Since $p \neq 0$, then $x = \dfrac{1}{\frac{1}{x}} = \dfrac{1}{\frac{p}{q}} = \dfrac{q}{p}$. Hence x can be written as a quotient of two integers with a nonzero denominator. Thus, x is rational.

1.7 pg 91 # 17

Show that if n is an integer and $n^3 + 5$ is odd, then n is even using

a a proof by contraposition

The contrapositive is "If n is odd, then $n^3 + 5$ is even." Assume that n is odd. We can now write $n = 2k + 1$ for some integer k. Then $n^3 + 5 = (2k + 1)^3 + 5 = 8k^3 + 12k^2 + 6k + 6 = 2(4k^3 + 6k^2 + 3k + 3)$. Thus $n^3 + 5$ is two times some integer, so it is even by the definition of an even integer.

b a proof by contradiction

Suppose that $n^3 + 5$ is odd and that n is odd. Since n is odd, the product of odd numbers is odd. So we can see that n^3 is odd. But if we subtract 5, then the difference between the two odd numbers $n^3 + 5$ and n^3 is even. Thus, our assumption was wrong and it is a contradiction.

1.7 pg 91 # 23

Show that at least ten of any 64 days chosen must fall on the same day of the week.

We will use proof by contradiction. Suppose that there were only nine or fewer days on each day of the week. This would mean that we can have at most $9 \cdot 7 = 63$ days we could have chosen. Since we picked 64 days, we have a contradiction because $63 \neq 64$. This means that at least ten of the days must be on the same day of the week.

1.7 pg 91 # 27

Prove that if n is a positive integer, then n is odd if and only if $5n + 6$ is odd.

We will use a direct proof on "If n is odd, then $5n + 6$ is odd". Assume n is odd, so $n = 2k + 1$ for some integer k. Then $5n+6 = 5(2k+1)+6 = 10k+5+6 = 10k+11 = 2(5k+5)+1$. Thus, $5n+6$ is odd. We now must prove the converse, "If $5n + 6$ is odd, then n is odd." For this, we will use proof by contrapositive. So the statement becomes "If n is not odd, then $5n+6$ is not odd." Assume that n is not odd, so $n = 2k$ for some integer k. Then $5n + 6 = 5(2k) + 6 = 10k + 6 = 2(5k + 3)$. Thus, $5n + 6$ is not odd. Hence we have proven that n is odd if and only if $5n + 6$ is odd.

Supplementary Exercises pg 113 # 39

Prove that if x is irrational and $x \geq 0$, then $\sqrt{x}$ is irrational.

We will use proof by contraposition. The contrapositive is "If $\sqrt{x}$ is rational, then x is rational" assuming that $x \geq 0$ for the whole statement. Suppose that $\sqrt{x} = p/q$ is rational and $q \neq 0$. Then $x = (\sqrt{x})^2 = p^2/q^2$ is also rational.