7.4 Expected Value and Variance

Problem

Suppose you roll a (fair) 6-sided die three times.

(a) Compute $E(X)$.

Let X_1, X_2, X_3 be random variables where X_i is 0 if the i^{th} roll is not a 6, and 1 if it is. Since $X = X_1 + X_2 + X_3$, we know that:

$$E(X) = E(X_1 + X_2 + X_3) = E(X_1) + E(X_2) + E(X_3).$$

Notice that for each of the X_is,

$$E(X_i) = 1/6 \cdot 1 + 5/6 \cdot 0 = 1/6$$

so $E(X) = 3/6 = 1/2$.

(b) Compute $E(X^2)$.

The possible values for X are 0, 1, 2, and 3, so the possible values for X^2 are 0, 1, 4, and 9. Consider this table to help us compute $E(X^2)$. Here, k is the number of die rolls showing 6, so $X^2(k) = k^2$. We can compute $p(X^2 = k)$ by considering three Bernoulli trials with probability of success $1/6$ and probability of failure $5/6$.

k	$X^2(k) = k^2$	$p(X^2 = k)$
0	0	$C(3,0)(1/6)^0(5/6)^3$
1	1	$C(3,1)(1/6)^1(5/6)^2$
2	4	$C(3,2)(1/6)^2(5/6)^1$
3	9	$C(3,3)(1/6)^3(5/6)^0$

Recall that from Rosen that:

$$E(Y) = \sum_{s \in S} p(s)Y(s)$$

where S is the sample space of all possible outcomes. Since we span S by considering each of the values of X^2, we can compute $E(X^2)$ using the above table:

$$\begin{aligned} E(X^2) &= (0 \cdot 1(5/6)^3) \\ &\quad + (1 \cdot 3(1/6)(5/6)^2) \\ &\quad + (4 \cdot 3(1/6)^2(5/6)) \\ &\quad + (9 \cdot 1(1/6)^3) \\ &= (3(1/6)(5/6)^2 + 12(1/6)^2(5/6) + 9(1/6)^3) \\ &= 2/3 \end{aligned}$$

(c) Compute $V(X)$.

$E(X) = 1/2$, $E(X)^2 = 1/4$, so:

$$V(X) = E(X^2) - E(X)^2 = 2/3 - 1/4 = 5/12.$$