7.2 Probability Theory

Assigning probabilities

Let S be a sample space of an experiment with a finite number of outcomes. We assign a probability $p(s)$ to each outcome s, so that $0 \leq p(s) \leq 1$ and $\sum_{s \in S} p(s) = 1$.

Probability Distribution

Let p be any function $p : S \to [0, 1]$ such that:

- $0 \leq p(s) \leq 1$
- $\sum_{s \in S} p(s) = 1$

Then p is called the *probability distribution*.
The probability under p of any event $E \subseteq S$ is just $p(E) = \sum_{s \in E} p(s)$.

Uniform Probability Distribution

Suppose that S is a set with n elements. The *uniform distribution* assigns the probability $1/n$ to each element of S.

The probability of the event E is the sum of the probabilities of the outcomes in E. That is, $p(E) = \sum_{s \in E} p(s)$.

Probabilities of Complements

$\sum_{s \in E} p(s) = 1 = p(E) + p(\overline{E})$

Probabilities of Unions of Events

If $E_1, E_2, \ldots$ is a sequence of pairwise disjoint events in a sample space S, then

$$p(\bigcup_i E_i) = \sum_i p(E_i)$$

Conditional Probability

Let E and F be events with $p(F) > 0$. The *conditional probability* of E given F, denoted by $p(E \mid F)$, is defined as

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)}$$

Independence

The events E and F are *independent* if and only if $p(E \cap F) = p(E)p(F)$.

Pairwise and Mutual Independence

The events $E_1, E_2, \ldots, E_n$ are pairwise independent if and only if $p(E_i \cap E_j) = p(E_i) \cdot p(E_j)$ for all pairs i and j with $i \leq j \leq n$.

The events are mutually independent if

$$p(E_{i_1} \cap E_{i_2} \cap \cdots E_{i_m}) = p(E_{i_1})p(E_{i_2}) \cdots p(E_{i_m})$$

whenever i_j, $j = 1, 2, \ldots, m$, are integers with $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ and $m \geq 2$.

Binomial Distribution

The probability of exactly k successes in n independent Bernoulli trials, with probability of success p and probability of failure $q = 1 - p$, is

$$C(n, k)p^k q^{n-k}$$

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Find the probability of each outcome when a biased die is rolled, if rolling a 2 or rolling a 4 is three times as likely as rolling each of the other four numbers on the die and it is equally likely to roll a 2 or a 4.

Let $p(1) = p(3) = p(5) = p(6) = c$
Then, $p(2) = p(4) = 3c$
Since $p(1) + p(2) + p(3) + p(4) = p(5) + p(6) = 1$
Then $c + 3c + c + 3c + c + c = 10c = 1$
Therefore, $c = \dfrac{1}{10}$

$p(1) = p(3) = p(5) = p(6) = \dfrac{1}{10}$
$p(2) = p(4) = \dfrac{3}{10}$

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What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up heads?

$p(F) =$ "first coin flip comes up heads" = $1/2$
This is either heads or tails.

$|E \cap F| =$ "we get four heads out of five coin flips, and the first flip comes up heads" $= C(4, 3)$.
Assume that the first coin flip is chosen for us, and so we only have $r = 3$ choices to make from $n = 4$ items. We then divide by the total number of possible outcomes, 2^5.

$p(E \mid F) = p(E \cap F)/p(F)$
$= \dfrac{C(4,3)/2^5}{1/2}$
$= \dfrac{C(4,3)}{2^4}$
$= \dfrac{4!/(3!1!)}{16}$
$= \dfrac{4}{16}$
$= \dfrac{1}{4}$

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Let E and F be the events that a family of n children has children of both sexes and has at most one boy, respectively. Are E and F independent if

a) $n = 2$?

We need to find $p(E), p(F)$, and $p(E \cap F)$. If $p(E) \cdot p(F) = p(E \cap F)$, then they are independent.

If a family has only two children, then there are 4 outcomes: BB, BG, GB, and GG.
$p(E) = 2/4$ because only two of the cases have both sexes.
$p(F) = 3/4$ because three of the cases have at most one boy.
$p(E \cap F) = 2/4$ because only two of the cases have both sexes and at most one boy.

$p(E) \cdot p(F) = 2/4 \cdot 3/4 = 6/16 = 3/8$
Since $3/8 \neq 2/4$, these events are not independent.

b) $n = 4$?

If a family has four children, then there are 16 outcomes.
$p(E) = 14/16$ because only 14 cases have both sexes.
$p(F) = 5/16$ because only 5 cases have at most one boy (GGGG, BGGG, GBGG, GGBG, GGGB)
$p(E \cap F) = 4/16$ because only four cases have both sexes and at most one boy (BGGG, GBGG, GGBG, GGGB)

$p(E) \cdot p(F) = 14/16 \cdot 5/16 = 70/256 = 35/128$
Since $35/128 \neq 4/16$, these events are not independent.

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A group of six people play the game of "odd person out" to determine who will buy refreshments. Each person flips a fair coin. If there is a person whose outcome is not the same as that of any other member of the group, this person has to buy the refreshments. What is the probability that there is an odd person out after the coins are flipped once?

We can model this problem using the binomial distribution. $C(n,k)p^k q^{n-k}$

$n = 6$ because we have six Bernoulli trials (6 coins being flipped).
$p = 1/2$ because is the probability of heads.
$q = 1 - p = 1 - 1/2 = 1/2$
There are only two cases when there is an odd one out, they are 5 heads and 1 tails or 1 heads and 5 tails. So, $k = 5$ or $k = 1$ for the number of heads.
For five heads and one tails:
$C(6, 5)(1/2)^5(1/2)^{6-5}$
$= (6!/(5!1!))(1/32)(1/2)$
$= 6/64 = 3/32$

For one heads and five tails:
$C(6, 1)(1/2)^1(1/2)^{6-1}$
$= (6!/(1!5!))(1/2)(1/32)$
$= 6/64 = 3/32$

Add them together and we get our probability for odd one out.
$3/32 + 3/32 = 6/32 = 3/16$