# 5.2 Strong Induction and Well-Ordering

### Strong Induction

To prove that $P(n)$ is true for all positive integers $n$, where $P(n)$ is a propositional function, complete two steps:

- **Basis Step:** Verify that the proposition $P(1)$ is true.
- **Inductive Step:** Show the conditional statement $[P(1) \land P(2) \land \cdots \land P(k)] \rightarrow P(k+1)$ is true for all positive integers $k$.

### Generalizing Strong Induction

- Handle cases where the inductive step is valid only for integers greater than a particular integer
  - $P(n)$ is true for $\forall n \geq b$ ($b$: fixed integer)
- **Basis Step**: Verify that $P(b), P(b+1), \ldots, P(b+j)$ are true ($j$: a fixed positive integer)
- **Inductive Step**: Show that the conditional statement $[P(b) \land P(b+1) \land \cdots \land P(k)] \rightarrow P(k+1)$ is true for all positive integers $k \geq b+j$

### 5.2 pg 341 # 3

Let $P(n)$ be the statement that a postage of $n$ cents can be formed using just 3-cent stamps and 5-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for $n \geq 8$.

a) Show that the statements $P(8), P(9)$, and $P(10)$ are true, completing the basis step of the proof.

   $8 = 3 \cdot 1 + 5 \cdot 1$
   $9 = 3 \cdot 3 + 5 \cdot 0$
   $10 = 3 \cdot 0 + 5 \cdot 2$

b) What is the inductive hypothesis of the proof?

   Any value $j$ ($8 \leq j \leq k$) where $k \geq 10$, can be expressed as $j = 3a + 5b$ with $a$ and $b$ being non-negative integers.

c) What do you need to prove in the inductive step?

   Assuming the inductive hypothesis, we want to show that we can express $k+1$ as $3a + 5b$ with $a$ and $b$ being nonnegative integers.

d) Complete the inductive step for $k \geq 10$.

   Since we want to show $P(k+1)$, we can use $P(k-2)$, which is true by inductive hypothesis since $8 \leq k-2 \leq k$.

$k - 2 = 3a + 5b$
$k - 2 + 3 = 3a + 4b + 3$
$k + 1 = 3(a + 1) + 5b$

Explanation:
Our base cases: 8, 9, and 10 can generate any integer value when a multiple of three is added.
e.g.
$8 + 3 = 11$
$9 + 3 = 12$
$10 + 3 = 13$

$8 + 6 = 14$
$9 + 6 = 15$
$10 + 6 = 16$
...
Therefore, by assuming $k-2$ and adding a 3-cent stamp, we can get to $k+1$ cents of postage.

e) Explain why these steps show that this statement is true whenever $n \geq 8$.

We have completed both the basis step and the inductive step, so by the principle of strong induction, the statement is true for every integer $n$ greater than or equal to 8.

**5.2 pg 342 # 7**

What amounts of money can be formed using just two-dollar bills and five-dollar bills? Prove your answer using strong induction.

2 dollars can also be formed, which can be proved separately.

$4 = 2 \cdot 2 + 5 \cdot 0$
$5 = 2 \cdot 0 + 5 \cdot 1$
$6 = 2 \cdot 3 + 5 \cdot 0$
$7 = 2 \cdot 1 + 5 \cdot 1$
$8 = 2 \cdot 4 + 5 \cdot 0$
$9 = 2 \cdot 2 + 5 \cdot 1$
$10 = 2 \cdot 5 + 5 \cdot 0$

Inductive hypothesis: $P(j)$ = any value $j$ ($4 \leq j \leq k$), can be expressed as $j = 2a + 5b$ with $a$ and $b$ being non-negative integers.

Basis Step: $P(4)$ and $P(5)$ are true (see above).

Inductive step:
Assume that for $5 \leq k$, $P(k-1)$ is true.
$k - 1 = 2a + 5b$

$k - 1 + 2 = 2a + 5b + 2$
$k + 1 = 2(a + 1) + 5b$
This completes the inductive step.

Therefore, by the principle of strong induction, $P(n)$ is true for all $n \geq 4$.
Explanation:
From $P(4)$ and $P(5)$, we can add a multiple of two (using 2-dollar bills) and reach any positive integer value $\geq 4$.

**5.2 pg 343 # 25**

Suppose that $P(n)$ is a propositional function. Determine for which positive integers $n$ the statement $P(n)$ must be true, and justify your answer, if

a) $P(1)$ is true; for all positive integers $n$, if $P(n)$ is true, then $P(n + 2)$ is true.

   $P(1)$ is true, so $P(1 + 2)$ is true, according to the statement.
   $P(3)$ is true, so $P(3 + 2)$ is true.
   $P(5)$ is true, so $P(5 + 2)$ is true.
   $P(n)$ is true when $n = 1, 3, 5, 7, 9, \ldots$

b) $P(1)$ and $P(2)$ are true; for all positive integers $n$, if $P(n)$ and $P(n + 1)$ are true, then $P(n + 2)$ is true.

   $P(1)$ and $P(1 + 1)$ are true, so $P(1 + 2)$ is true too.
   $P(2)$ and $P(2 + 1)$ are true, so $P(2 + 2)$ is true too.
   $P(3)$ and $P(3 + 1)$ are true, so $P(3 + 2)$ is true too.
   $P(4)$ and $P(4 + 1)$ are true, so $P(4 + 2)$ is true too.
   $P(n)$ is true when $n$ is any positive integer.

c) $P(1)$ is true; for all positive integers $n$, if $P(n)$ is true, then $P(2n)$ is true.

   $P(1)$ is true, so $P(2 \cdot 1)$ is true.
   $P(2)$ is true, so $P(2 \cdot 2)$ is true.
   $P(4)$ is true, so $P(2 \cdot 4)$ is true.
   $P(8)$ is true, so $P(2 \cdot 8)$ is true.
   $P(n)$ is true when $n$ is an integer and a power of 2. (i.e. $n = 2, 4, 8, 16, \ldots$)