2.4 Sequences and Summations

Sequences

Sequences are ordered lists of elements.

Geometric Progression

A *geometric progression* is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n$$

where the initial term a and the common ratio r are real numbers.

Arithmetic Progression

An *arithmetic progression* is a sequence of the form

$$a, a + d, a + 2d, \ldots, a + nd$$

where the initial term a and the common difference d are real numbers.

Recurrence Relation

A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$ is a nonnegative integer. A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.

Fibonacci Sequence

The *Fibonacci sequence*, $f_0, f_1, f_2, \ldots$, is defined by the initial condition $f_0 = 0, f_1 = 1$, and the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

for $n = 2, 3, 4, \ldots$.

Summations

$\sum_{i=k}^{n} a_i$ means "$a_k + a_{k+1} + a_{k+2} + a_{k+3} + \ldots + a_n$"
for each i from k to n, find a_i and sum the results.

Common Summations and Closed Forms

Sum	Closed Form
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r-1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x\rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x\rvert < 1$	$\frac{1}{(1-x)^2}$

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Find these terms of the sequence $\{a_n\}$, where $a_n = 2 \cdot (-3)^n + 5^n$.

a) a_0

$2 \cdot (-3)^0 + 5^0 = 3$

b) a_1

$2 \cdot (-3)^1 + 5^1 = -1$

c) a_4

$2 \cdot (-3)^4 + 5^4 = 2 \cdot 81 + 625 = 162 + 625 = 787$

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Is the sequence $\{a_n\}$ a solution of the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$ if

a) $a_n = 0$?

To solve these problems, we need to substitute the value of a_n into the recurrence relation and see if they are equal.
$0 = 8 \cdot 0 - 16 \cdot 0$
$= 0$
Yes

b) $a_n = 1$?

$1 = 8 \cdot 1 - 16 \cdot 1$
$= 8 - 16$
$= -8$
No.

c) $a_n = 2^n$?

$2^n = 8 \cdot 2^{n-1} - 16 \cdot 2^{n-2}$
$= 8 \cdot 2 \cdot 2^{n-2} - 16 \cdot 2^{n-2}$
$= 2^{n-2}(8 * 2 - 16)$
$= 2^{n-2}(16 - 16)$
$= 2^{n-2}(0)$
$= 0$
No.

d) $a_n = 4^n$?

$4^n = 8 \cdot 4^{n-1} - 16 \cdot 4^{n-2}$
$= 8 \cdot 4 \cdot 4^{n-2} - 16 \cdot 4^{n-2}$
$= 4^{n-2}(8 \cdot 4 - 16)$
$= 4^{n-2}(8 \cdot 4 - 16)$
$= 4^{n-2}(32 - 16)$
$= 4^{n-2}(16)$
$= 4^{n-2} \cdot 4^2$
$= 4^n$
Yes.

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Find the solution to each of these recurrence relations and initial conditions. Use an iterative approach.

a) $a_n = 3a_{n-1}, a_0 = 2$

$a_1 = a_0 \cdot 3 = (2) \cdot 3$
$a_2 = a_1 \cdot 3 = (2 \cdot 3) \cdot 3$
$a_3 = a_2 \cdot 3 = (2 \cdot 3 \cdot 3) \cdot 3$
$\ldots$
$a_n = 3 \cdot a_{n-1} = 2 \cdot 3^n$

In a_1, a_2, a_3, we see that the number of times we multiply by three is equal to the value of our subscript. We also see that a_0 is included once in each of our terms. so, $a_n = 2 \cdot 3^n$. **Note:** Since we can express this relation in the form $a \cdot r^n$, it is a geometric progression.

b) $a_n = a_{n-1} + 2, a_0 = 3$

$a_1 = a_0 + 2 = (3) + 2$
$a_2 = a_1 + 2 = (3 + 2) + 2$
$a_3 = a_2 + 2 = (3 + 2 + 2) + 2$
$\ldots$
$a_n = a_{n-1} + 2 = 3 + 2n$

Again, in a_1, a_2, a_3, we see that the number of times we add two is equal to the value of our subscript. And again a_0 is included once in each of our terms. So, $a_n = 3 + 2 \cdot n$. **Note:** Since we can express this relation in the form $a + d \cdot n$, it is an arithmetic progression.

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Suppose that the number of bacteria in a colony triples every hour.

a) Set up a recurrence relation for the number of bacteria after n hours have elapsed.

Since the number of bacteria at n hours is three times the bacteria at $n-1$ hours, our recurrence relation is $a_n = 3a_{n-1}$

b) If 100 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?

We first solve the recurrence relation by iteration.
$a_1 = 3 \cdot a_0 = 3 \cdot (100)$
$a_2 = 3 \cdot a_1 = 3 \cdot (3 \cdot 100)$
$a_3 = 3 \cdot a_2 = 3 \cdot (3 \cdot 3 \cdot 100)$
$\ldots$
$a_n = 3 \cdot a_{n-1} = 3^n \cdot 100$
So, $a_{10} = 3^{10} \cdot 100 = 5,904,900$

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What are the values of these sums?

a) $\sum_{k=1}^{5}(k+1)$
$= (1+1) + (2+1) + (3+1) + (4+1) + (5+1)$
$= 2 + 3 + 4 + 5 + 6$
$= 20$

d) $\sum_{j=0}^{8}(2^{j+1} - 2^j)$
$= (2^{0+1} - 2^0) + (2^{1+1} - 2^1) + (2^{2+1} - 2^2) + (2^{3+1} - 2^3) + (2^{4+1} - 2^4) + (2^{5+1} - 2^5) + (2^{6+1} - 2^6) + (2^{7+1} - 2^7) + (2^{8+1} - 2^8)$
$= (2^1 - 2^0) + (2^2 - 2^1) + (2^3 - 2^2) + (2^4 - 2^3) + (2^5 - 2^4) + (2^6 - 2^5) + (2^7 - 2^6) + (2^8 - 2^7) + (2^9 - 2^8)$
$= -2^0 + 2^9$
$= -1 + 512$
$= 511$

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Compute each of these double sums.

a) $\sum_{i=1}^{2}\sum_{j=1}^{3}(i+j)$

$= \sum_{i=1}^{2}((i+1)+(i+2)+(i+3))$
$= \sum_{i=1}^{2}(3i+6)$
$= (3(1)+6)+(3(2)+6)$
$= 3+6+6+6$
$= 21$

c) $\sum_{i=1}^{3}\sum_{j=0}^{2} i$

$= \sum_{i=1}^{3}(i+i+i)$
$= (1+1+1)+(2+2+2)+(3+3+3)$
$= 3+6+9$
$= 18$

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Find $\sum_{k=100}^{200} k$.

$\sum_{k=100}^{200} k$
$= \sum_{k=1}^{200} k - \sum_{k=1}^{99} k$
$= \frac{200(200+1)}{2} - \frac{99(99+1)}{2}$
$= \frac{200(201)}{2} - \frac{99(100)}{2}$
$= \frac{40200}{2} - \frac{9900}{2}$
$= 20100 - 4950$
$= 15150$