8.1 Applications of Recurrence Relations

Recurrence Relation:
A recurrence relation is an equation that recursively defines a sequence, once one or more initial terms are given: each further term of the sequence is defined as a function of the preceding terms. - Wikipedia

8.1 pg. 510 # 3

A vending machine dispensing books of stamps accepts only one-dollar coins, \$1 bills, and \$5 bills.

a Find a recurrence relation for the number of ways to deposit n dollars in the vending machine, where the order in which the coins and bills are deposited matters.

Let a_n be the number of ways to deposit n dollars. Consider depositing a one-dollar coin, then we have $n-1$ dollars left to deposit. This will give us a_{n-1} ways to deposit n dollars. Depositing \$1 bills will also be the same as depositing one-dollar coins, so this will also give us a_{n-1} ways to deposit n dollars. Lastly, if we deposit a \$5 bill, then we have $n-5$ dollars left to deposit, so we have a_{n-5} ways to deposit n dollars. This would mean that our recurrence relation is $a_n = a_{n-1} + a_{n-1} + a_{n-5}$. Simplify the equation and we get $a_n = 2a_{n-1} + a_{n-5}$. Note that this is only valid when $n \geq 5$.

b What are the initial conditions?

The initial conditions are the different ways to deposit n dollars up to $n = 4$. So, $a_0 = 1$ because there is only one way to deposit 0 dollars (do nothing). To find a_1, we know that there are two ways to deposit 1 dollar, by using the dollar coin or the \$1 bill, so $a_1 = 2$. To find a_2, we know there are 2^2 ways to deposit 2 dollars, so $a_2 = 4$. Following the same line of thinking, we know $a_3 = 8$ and $a_4 = 16$.

c How many ways are there to deposit \$10 for a book of stamps?

We need to find a_{10}, so we will have to work our way up from a_5.
$a_5 = 2a_4 + a_0 = 2(16) + 1 = 33$
$a_6 = 2a_5 + a_1 = 2(33) + 2 = 68$
$a_7 = 2a_6 + a_2 = 2(68) + 4 = 140$
$a_8 = 2a_7 + a_3 = 2(140) + 8 = 288$
$a_9 = 2a_8 + a_4 = 2(288) + 16 = 592$
$a_{10} = 2a_9 + a_5 = 2(592) + 33 = 1217$
There are 1217 ways to deposit \$10.

8.1 pg. 510 # 7

a Find a recurrence relation for the number of bit strings of length n that contain a pair of consecutive 0s.

Let a_n be the number of bit strings of length n that contains a pair of consecutive 0s. Consider that the first bit of the bit string is 1, then the remaining $n-1$ bits of the string would contain the pair of consecutive 0s. Let us also consider if the start of the bit string was 01, then the remaining $n-2$ bits of the string would contain the pair of consecutive 0s. Lastly, if our bit string started 00, then it follows that any string of $n-2$ would satisfy the condition. This would mean that our recurrence relation is $a_n = a_{n-1} + a_{n-2} + 2^{n-2}$, valid for $n \geq 2$. Remember that there are 2^n bit strings of length n.

b What are the initial conditions?

Since there are no bit strings of length 0 and 1 that contains two consecutive 0s, we know $a_0 = 0$ and $a_1 = 0$.

c How many bit strings of length seven contain two consecutive 0s?

Compute a_7 starting from a_2.
$a_2 = a_1 + a_0 + 2^0 = 0 + 0 + 1 = 1$
$a_3 = a_2 + a_1 + 2^1 = 1 + 0 + 2 = 3$
$a_4 = a_3 + a_2 + 2^2 = 3 + 1 + 4 = 8$
$a_5 = a_4 + a_3 + 2^3 = 8 + 3 + 8 = 19$
$a_6 = a_5 + a_4 + 2^4 = 19 + 8 + 16 = 43$
$a_7 = a_6 + a_5 + 2^5 = 43 + 19 + 32 = 94$
There are 94 bit strings of length seven that contains a pair of consecutive 0s.

8.1 pg. 511 # 19

Messages are transmitted over a communications channel using two signals. The transmittal of one signal requires 1 microsecond, and the transmittal of the other signal requires 2 microseconds.

a Find a recurrence relation for the number of different messages consisting of sequences of these two signals, where each signal in the message is immediately followed by the next signal, that can be sent in n microseconds.

Let a_n be the number of different messages that can be sent in n microseconds. To send a message, we can first start off with a signal that is 1 microsecond and then send the remaining $n-1$ microseconds. We can also start off with a signal that is 2 microseconds and then send the remaining $n-2$ microseconds. Then it follows that our recurrence relation is $a_n = a_{n-1} + a_{n-2}$, for $n \geq 2$.

b What are the initial conditions?

There is only 1 way to send a 0 microsecond message, and that is to send nothing. So, $a_0 = 1$. To send 1 microsecond, it is obvious that there is only one way to accomplish this, so $a_1 = 1$.

c How many different messages can be sent in 10 microseconds using these two signals?

Compute a_{10}.
$a_2 = a_1 + a_0 = 1 + 1 = 2$
$a_3 = a_2 + a_1 = 2 + 1 = 3$
$a_4 = a_3 + a_2 = 3 + 2 = 5$
$a_5 = a_4 + a_3 = 5 + 3 = 8$

$a_6 = a_5 + a_4 = 8 + 5 = 13$
$a_7 = a_6 + a_5 = 13 + 8 = 21$
$a_8 = a_7 + a_6 = 21 + 13 = 34$
$a_9 = a_8 + a_7 = 34 + 21 = 55$
$a_{10} = a_9 + a_8 = 55 + 34 = 89$
There are 89 messages that are 10 microseconds long.

8.1 pg. 512 # 27

a Find a recurrence relation for the number of ways to lay out a walkway with slate tiles if the tiles are red, green, or gray, so that no two red tiles are adjacent and tiles of the same color are considered indistinguishable.

We assume that the walkway is one tile in width and n tiles long, from start to finish. Let a_n be the number of ways to lay out a walkway with n tiles. Consider that the first tile is green, then there are a_{n-1} ways to lay out the remaining walkway with $n-1$ tiles without adjacent red tiles. Also, if the first tile is gray, then it also follows that there are a_{n-1} ways to lay out the remaining walkway with $n-1$ tiles. We can also consider the first tile is red followed by a green tile, then there are a_{n-2} ways to lay out the remaining walkway with $n-2$ tiles. Continuing from before, red followed by grey would also leave us to lay out the rest of the walkway with $n-2$ tiles. This would mean that our recurrence relation is $a_n = a_{n-1} + a_{n-1} + a_{n-2} + a_{n-2}$. Simplified, $a_n = 2a_{n-1} + 2a_{n-2}$, for $n \geq 2$.

b What are the initial conditions for the recurrence relation in part (a)?

$a_0 = 1$ because there is only one way to lay no tiles. $a_1 = 3$ because we can choose any color tile for laying down 1 tile.

c How many ways are there to lay out a path of seven tiles as described in part (a)?

Compute a_7.
$a_2 = 2a_1 + 2a_0 = 2(3) + 2(1) = 8$
$a_3 = 2a_2 + 2a_1 = 2(8) + 2(3) = 22$
$a_4 = 2a_3 + 2a_2 = 2(22) + 2(8) = 60$
$a_5 = 2a_4 + 2a_3 = 2(60) + 2(22) = 164$
$a_6 = 2a_5 + 2a_4 = 2(164) + 2(60) = 448$
$a_7 = 2a_6 + 2a_5 = 2(448) + 2(164) = 1224$
There are 1224 ways to lay down seven tiles.