8.2 Solving Linear Recurrence Relations

- Determine if recurrence relation is homogeneous or nonhomogeneous.
- Determine if recurrence relation is linear or nonlinear.
- Determine whether or not the coefficients are all constants.
- Determine what is the degree of the recurrence relation.
- Need to know the general solution equations.
- Need to find characteristic equation.
- Need to find characteristic roots (can use determinant to help).

Determinants (optional)

When finding characteristic roots and determining which general solution to use for a recurrence relation of degree 2, using determinants can be helpful. From the quadratic equation, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the determinant is $b^2 - 4ac$.

Case 1: $b^2 - 4ac > 0$

You have two distinct real roots, r_1 and r_2, your general solution is $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$. (Theorem 1)

Case 2: $b^2 - 4ac = 0$

You have one root with multiplicity 2, r_0, your general solution is $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$. (Theorem 2)

Summary of general solutions

Theorem	Degree	Characteristic Roots	General Solution
1	2	$r_1 \neq r_2$	$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$
2	2	r_0 with multiplicity 2	$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$
3	k	k distinct roots $r_1, r_2, ..., r_k$	$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + ... + \alpha_k r_k^n$
4	k	t distinct roots $(r_1, r_2, ...r_t)$ with multiplicities $(m_1, m_2, ..., m_t)$	$a_n = (\alpha_{1,0} + \alpha_{1,1} n + ... + \alpha_{1,m_1-1} n^{m_1-1}) \cdot r_1^n + (\alpha_{2,0} + \alpha_{2,1} n + ... + \alpha_{2,m_2-1} n^{m_2-1}) \cdot r_2^n + ... + (\alpha_{t,0} + \alpha_{t,1} n + ... + \alpha_{t,m_t-1} n^{m_t-1}) \cdot r_t^n$

Nonhomogenous recurrence relations

Theorem 5: If $a_n^{(p)}$ is a particular solution to the linear nonhomogeneous recurrence relation with constant coefficients, $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} + F(n)$, then every solution is of the form $a_n^{(p)} + a_n^{(h)}$ where $a_n^{(h)}$ is a solution of the associated homogeneous recurrence relation, $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$.

Theorem 6: Assume that a_n satisfies the linear nonhomogeneous recurrence relation with constant coefficients:
$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} + F(n)$
with $F(n)$ of the form:
$F(n) = (b_t n^t + b_{t-1} n^{t-1} + ... + b_1 n + b_0) s^n$
where $b_0, b_1, ..., b_t$ and s are real numbers.

Case 1: If s is not a characteristic root of the associated linear homogeneous recurrence relation with constant coefficients, there is a particular solution of the form

$(\alpha_t n^t + \alpha_{t-1} n^{t-1} + ... + \alpha_1 n + \alpha_0) s^n$

Case 2: If s is a characteristic root of multiplicity m, there is a particular solution of the form

$n^m(\alpha_t n^t + \alpha_{t-1} n^{t-1} + ... + \alpha_1 n + \alpha_0) s^n$

8.2 pg. 524 # 1

Determine which of these are linear homogeneous recurrence relations with constant coefficients. Also, find the degree of those that are.

a $a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$

Yes. Degree 3.

b $a_n = 2na_{n-1} + a_{n-2}$

No. $2n$ is not a constant coefficient.

c $a_n = a_{n-1} + a_{n-4}$

Yes. Degree 4.

d $a_n = a_{n-1} + 2$

No. This is nonhomogeneous because of the 2.

e $a_n = a_{n-1}^2 + a_{n-2}$

No. This is not linear because of a_{n-1}^2.

f $a_n = a_{n-2}$

Yes. Degree 2.

g $a_n = a_{n-1} + n$

No. This is nonhomogeneous because of the n.

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Solve these recurrence relations together with the initial conditions given.

a $a_n = 2a_{n-1}$ for $n \geq 1, a_0 = 3$

Characteristic equation: $r - 2 = 0$
Characteristic root: $r = 2$
By using Theorem 3 with $k = 1$, we have $a_n = \alpha 2^n$ for some constant α. To find α, we can use the initial condition, $a_0 = 3$, to find it.
$3 = \alpha 2^0$
$3 = \alpha 1$
$3 = \alpha$

So our solution to the recurrence relation is $a_n = 3 \cdot 2^n$.

b $a_n = a_{n-1}$ for $n \geq 1, a_0 = 2$

Same as problem (a).
Characteristic equation: $r - 1 = 0$
Characteristic root: $r = 1$
Use Theorem 3 with $k = 1$ like before, $a_n = \alpha 1^n$ for some constant α.
Find α.
$2 = \alpha 1^0$
$2 = \alpha$

So the solution is $a_n = 2 \cdot 1^n$. But we can simplify this since $1^n = 1$ for any n, so our solution is $a_n = 2$ for any n.

c $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2, a_0 = 1, a_1 = 0$

Characteristic equation: $r^2 - 5r + 6 = 0$
Determinant: $5^2 - 4(1)(6) = 25 - 24 = 1$
Since our determinant is greater than 0, we know we can use Theorem 1.
Find the characteristic root.
We can factor $r^2 - 5r + 6 = 0$ into $(r - 2)(r - 3) = 0$.
So our roots are $r_1 = 2$ and $r_2 = 3$.
Use Theorem 1 to find our general solution: $a_n = \alpha_1 2^n + \alpha_2 3^n$ for some constants α_1 and α_2.
Find α_1 and α_2 by using our initial conditions.
For $a_0 = 1$
$1 = \alpha_1 2^0 + \alpha_2 3^0$
$1 = \alpha_1 + \alpha_2$
For $a_1 = 0$
$0 = \alpha_1 2^1 + \alpha_2 3^1$
$0 = 2\alpha_1 + 3\alpha_2$

Solve the system of equations:
$\alpha_1 = 1 - \alpha_2$

$0 = 2(1 - \alpha_2) + 3\alpha_2$
$0 = 2 - 2\alpha_2 + 3\alpha_2$
$0 = 2 + \alpha_2$
$\alpha_2 = -2$
$\alpha_1 = 1 - (-2)$
$\alpha_1 = 3$

Our solution is $a_n = 3 \cdot 2^n - 2 \cdot 3^n$

d $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2, a_0 = 6, a_1 = 8$

Characteristic equation: $r^2 - 4r + 4 = 0$.
Determinant: $16 - 4(1)(4) = 16 - 16 = 0$.
Determinant is 0, we can use Theorem 2.
Find the characteristic root.
Factor $r^2 - 4r + 4 = 0$ to $(r - 2)^2 = 0$.
Our root is $r = 2$ with multiplicity 2.
Use Theorem 2 to find our general solution: $a_n = \alpha_1 2^n + \alpha_2 n 2^n$ for some constants α_1 and α_2.
Find α_1 and α_2 by using our initial conditions.
For $a_0 = 6$
$6 = \alpha_1 2^0 + \alpha_2 \cdot 0 \cdot 2^0$
$6 = \alpha_1$
For $a_1 = 8$
$8 = \alpha_1 2^1 + \alpha_2 \cdot 1 \cdot 2^1$
$8 = 2\alpha_1 + 2\alpha_2$

Solve the system of equations:
$8 = 2(6) + 2\alpha_2$
$8 = 12 + 2\alpha_2$
$-4 = 2\alpha_2$
$\alpha_2 = -2$

Our solution is $a_n = 6 \cdot 2^n - 2n \cdot 2^n = (6 - 2n)2^n$.

e $a_n = -4a_{n-1} - 4a_{n-2}$ for $n \geq 2, a_0 = 0, a_1 = 1$

Characteristic equation: $r^2 + 4r + 4 = 0$.
We can factor $r^2 + 4r + 4 = 0$ into $(r + 2)^2 = 0$
So our characteristic roots are -2 with multiplicity 2.
Use Theorem 2: $a_n = \alpha_1(-2)^n + \alpha_2 n(-2)^n$ for some constants α_1 and α_2.
Find α_1 and α_2
For $a_0 = 0$
$0 = \alpha_1(-2)^0 + \alpha_2 0(-2)^0$
$\alpha_1 = 0$
For $a_1 = 1$
$1 = \alpha_1(-2)^1 + \alpha_2 \cdot 1 \cdot (-2)^1$
$1 = -2\alpha_1 - 2\alpha_2$

Solve the systems of equation and you get $\alpha_1 = 0$ and $\alpha_2 = -1/2$.
Our solution is $a_n = (-1/2)n(-2)^n = n(-2)^{n-1}$.

f $a_n = 4a_{n-2}$ for $n \geq 2, a_0 = 0, a_1 = 4$

Characteristic equation: $r^2 - 4 = 0$.
Factor the equation and you get $(r-2)(r+2) = 0$.
The characteristic roots are $r_1 = 2$ and $r_2 = -2$.
Use Theorem 1: $a_n = \alpha_1 2^n + \alpha_2(-2)^n$.
By using initial conditions, you'll get $0 = \alpha_1 + \alpha_2$ and $4 = 2\alpha_1 - 2\alpha_2$. When solved, $\alpha_1 = 1$ and $\alpha_2 = -1$
So the solution is $a_n = 2^n - (-2)^n$.

g $a_n = a_{n-2}/4$ for $n \geq 2, a_0 = 1, a_1 = 0$

Characteristic equation: $r^2 - 1/4 = 0$.
Factor to get $(r - 1/2)(r + 1/2) = 0$.
the characteristic roots are $r_1 = 1/2$ and $r_2 = -1/2$.
Use Theorem 1: $a_n = \alpha_1(1/2)^n + \alpha_2(-1/2)^n$.
By using initial conditions, you'll get $1 = \alpha_1 + \alpha_2$ and $0 = \alpha_1/2 - \alpha_2/2$. When solved, $\alpha_1 = 1/2$ and $\alpha_2 = 1/2$.
So the solution is $a_n = (1/2)(1/2)^n + (1/2)(-1/2)^n$.

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Find the solution to $a_n = 7a_{n-2} + 6a_{n-3}$ with $a_0 = 9, a_1 = 10, a_2 = 32$.

Characteristic equation: $r^3 - 7r - 6 = 0$.
This is a 3rd degree equation, so we have to use Theorem 3 or 4.
Need to find roots. Use rational root test.
We know our possible roots are $\pm 1, \pm 2, \pm 3, \pm 6$.
Test to see which of these number will be our root for the equation.
Test -1.
$(-1)^3 - 7(-1) - 6 = -1 + 7 - 6 = 0$ Good! We know -1 is a root.
Factor out $(r - (-1))$ in $r^3 - 7r - 6 = 0$ to get $(r+1)(r^2 - r - 6) = 0$. Continue factoring and we get $(r+1)(r-3)(r+2) = 0$.
We know our characteristic roots: $r_1 = -1, r_2 = 3, r_3 = -2$.
Our general solution using Theorem 3 is: $a_n = \alpha_1(-1)^n + \alpha_2 3^n + \alpha_3(-2)^n$.
Find $\alpha_1, \alpha_2, \alpha_3$ by using the initial conditions.
For $a_0 = 9$
$9 = \alpha_1(-1)^0 + \alpha_2 3^0 + \alpha_3(-2)^0$
$9 = \alpha_1 + \alpha_2 + \alpha_3$
For $a_1 = 10$
$10 = \alpha_1(-1)^1 + \alpha_2 3^1 + \alpha_3(-2)^1$
$10 = -\alpha_1 + 3\alpha_2 - 2\alpha_3$
For $a_2 = 32$
$32 = \alpha_1(-1)^2 + \alpha_2 3^2 + \alpha_3(-2)^2$

$32 = \alpha_1 + 9\alpha_2 + 4\alpha_3$

Solve the system of equations:
$9 = \alpha_1 + \alpha_2 + \alpha_3$
$10 = -\alpha_1 + 3\alpha_2 - 2\alpha_3$
$32 = \alpha_1 + 9\alpha_2 + 4\alpha_3$

Add the first equation and second equation together to get:
$19 = 4\alpha_2 - \alpha_3$
Add the second equation and third equation together to get:
$42 = 12\alpha_2 + 2\alpha_3$
Multiply $19 = 4\alpha_2 - \alpha_3$ by 2 and add with $41 = 12\alpha_2 + 2\alpha_3$ to get:
$80 = 20\alpha_2$
$4 = \alpha_2$
Substitute α_2 back in.
$19 = 4(4) - \alpha_3$
$19 = 16 - \alpha_3$
$-3 = \alpha_3$
With α_2 and α_3, we can find α_1
$9 = \alpha_1 + 4 - 3$
$9 = \alpha_1 + 1$
$8 = \alpha_1$

Our solution is $a_n = 8(-1)^n + 4(3)^n + (-3)(-2)^n$.

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What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has roots 1, 1, 1, 1, -2, -2, -2, 3, 3, -4?

Use Theorem 4 with 4 roots that have multiple multiplicities.
$r_1 = 1$ with multiplicity 4, $r_2 = -2$ with multiplicity 3, $r_3 = 3$ with multiplicity 2, and $r_4 = -4$ with multiplicity 1.
So our general solution is of the form:
$a_n = (\alpha_{1,0} + \alpha_{1,1}n + \alpha_{1,2}n^2 + \alpha_{1,3}n^3)(1)^n + (\alpha_{2,0} + \alpha_{2,1}n + \alpha_{2,2}n^2)(-2)^n + (\alpha_{3,0} + \alpha_{3,1}n)(3)^n + (\alpha_{4,0})(-4)^n$

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What is the general form of the particular solution guaranteed to exist by Theorem 6 of the linear nonhomogeneous recurrence relation $a_n = 8a_{n-2} - 16a_{n-4} + F(n)$ if

a $F(n) = n^3$?

We first need to find the associated homogeneous recurrence relation.
Associated homogeneous recurrence relation: $a_n = 8a_{n-2} - 16a_{n-4}$.

Characteristic equation: $r^4 - 8r^2 + 16 = 0$
Factor to find roots.
$r^4 - 8r^2 + 16 = 0$
$(r^2 - 4)(r^2 - 4) = 0$
$(r + 2)(r - 2)(r + 2)(r - 2) = 0$
Our roots are $r_1 = 2$ with multiplicity 2 and $r_2 = -2$ with multiplicity 2.

By Theorem 6, we know that in s^n, $s = 1$. Since 1 is not a root, we know the particular solution is of the form $(p_3n^3 + p_2n^2 + p_1n + p_0)1^n$. Simplified, $p_3n^3 + p_2n^2 + p_1n + p_0$.

b $F(n) = (-2)^n$?

Using Theorem 6, we know that $s = -2$ and -2 is a root with multiplicity 2, so the particular solution is of the form $n^2p_0(-2)^n$.

c $F(n) = n2^n$?

Using Theorem 6, $s = 2$ and 2 is a root with multiplicity 2, so the particular solution is of the form $n^2(p_1n + p_0)2^n$.

d $F(n) = n^24^n$?

Using Theorem 6, $s = 4$ and 4 is not a root, so the particular solution is of the form $(p_2n^2 + p_1n + p_0)4^n$.

e $F(n) = (n^2 - 2)(-2)^n$?

Using Theorem 6, $s = -2$ and -2 is a root with multiplicity 2, so the particular solution is of the form $n^2(p_2n^2 + p_1n + p_0)(-2)^n$.

f $F(n) = n^42^n$?

Using Theorem 6, $s = 2$ and 2 is a root with multiplicity 2, so the particular solution is of the form $n^2(p_4n^4 + p_3n^3 + p_2n^2 + p_1n + p_0)2^n$.

g $F(n) = 2$?

Using Theorem 6, $s = 1$ and 1 is not a root, so the particular solution is of the form p_0.

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a Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 3^n$.

The associated homogeneous recurrence relation is $a_n = 2a_{n-1}$.
The characteristic equation is $r - 2 = 0$.
Since our characteristic root is $r = 2$, we know by Theorem 3 that $a_n^{(h)} = \alpha 2^n$.
Note that $F(n) = 3^n$, so we know by Theorem 6 that $s = 3$ and 3 is not a root, the particular solution is of the form $a_n^{(p)} = c \cdot 3^n$. Plug $a_n^{(p)} = c \cdot 3^n$ into the recurrence relation and you'll get $c \cdot 3^n = 2c \cdot 3^{n-1} + 3^n$.
Simplify $c \cdot 3^n = 2c \cdot 3^{n-1} + 3^n$:
$c \cdot 3 = 2c + 3$

$3c = 2c + 3$
$c = 3$

Therefore, the particular solution that we seek is $a_n^{(p)} = 3 \cdot 3^n = 3^{n+1}$.
So the general solution is the sum of the homogeneous solution and the particular solution:
$a_n = \alpha 2^n + 3^{n+1}$.

b Find the solution of the recurrence relation in part (a) with initial condition $a_1 = 5$.

Plug the initial condition in and solve.
$5 = \alpha 2^1 + 3^{1+1}$
$5 = 2\alpha + 9$
$-4 = 2\alpha$
$\alpha = -2$

So the solution is $a_n = -2 \cdot 2^n + 3^{n+1} = -2^{n+1} + 3^{n+1}$.

Need to check answer!
Let's check a_2.
By the recurrence relation, we know $a_2 = 2a_1 + 3^2 = 2(5) + 9 = 19$.
By the solution, we know $a_2 = -2^{2+1} + 3^{2+1} = -2^3 + 3^3 = -8 + 27 = 19$
Since they both agree, we can be fairly confident that the answer is correct.

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Find all solutions of the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + (n + 1)2^n$.

Associated homogeneous recurrence relation is $a_n = 4a_{n-1} - 4a_{n-2}$.
Characteristic equation: $r^2 - 4r + 4 = 0$
Factor.
$r^2 - 4r + 4 = 0$
$(r - 2)^2 = 0$
Characteristic root is $r_0 = 2$ with multiplicity 2.
By Theorem 2, $a_n^{(h)} = \alpha 2^n + \beta n 2^n$.
Note that $F(n) = (n + 1)2^n$, so we know by Theorem 6 that $s = 2$ and 2 is a root with multiplicity 2, the particular solution is of the form $a_n^{(p)} = n^2(cn + d)2^n$.
Plug $a_n^{(p)} = n^2(cn+d)2^n$ into the recurrence relation and you'll get $n^2(cn+d)2^n = 4(n-1)^2(cn+d-c)2^{n-1} - 4(n-2)^2(cn+d-2c)2^{n-2} + (n+1)2^n$.
Simplify.
$n^2(cn + d)2^n = 4(n - 1)^2(cn + d - c)2^{n-1} - 4(n - 2)^2(cn + d - 2c)2^{n-2} + (n + 1)2^n$
$n^2(cn + d)2^n = (n - 1)^2(cn + d - c)2^{n+1} - (n - 2)^2(cn + d - 2c)2^n + (n + 1)2^n$
$n^2(cn + d) = (n - 1)^2(cn + d - c)2 - (n - 2)^2(cn + d - 2c) + (n + 1)$
$cn^3 + dn^2 = 2(n^2 - 2n + 1)(cn + d - c) - (n^2 - 4n + 4)(cn + d - 2c) + (n + 1)$
$cn^3+dn^2 = 2cn^3-6cn^2+2dn^2+6cn-4dn-2c+2d-cn^3+6cn^2-dn^2-12cn+4dn+8c-4d+n+1$
$cn^3 + dn^2 = cn^3 + dn^2 - 6cn + 6c - 2d + n + 1$
$cn^3 + dn^2 = cn^3 + dn^2 - 6cn + n + 6c - 2d + 1$
$cn^3 + dn^2 = cn^3 + dn^2 + n(-6c + 1) + (6c - 2d + 1)$
Solve $-6c + 1 = 0$ to find c.

$c = 1/6$.
Plug $c = 1/6$ into $6c - 2d + 1 = 0$ to find d.
$6(1/6) - 2d + 1 = 0$
$1 - 2d + 1 = 0$
$-2d = -2$
$d = 1$

Therefore the particular solution that we seek is $a_n^{(p)} = n^2(n/6 + 1)2^n$.
So the general solution is the sum of the homogeneous solution and particular solution:
$a_n = \alpha 2^n + \beta n 2^n + n^2 \cdot 2^n + n^3/6 \cdot 2^n = (\alpha + \beta n + n^2 + n^3/6)2^n$.