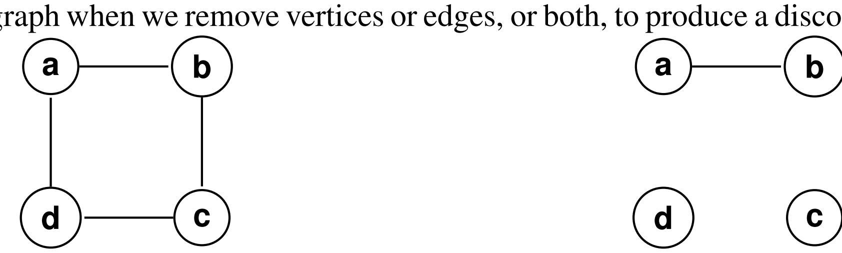
10.4 Connectivity

Path

- Let n be a nonnegative integer and G an undirected graph. A *path of length* n from u to v in G is a sequence of n edges $e_1, \ldots, e_n$ of G for which there exists a sequence $x_0 = u, x_1, \ldots, x_{n-1}, x_n = v$ of vertices such that e_i has, for $i = 1, \ldots, n$, the endpoints x_{i-1} and x_i.
- When the graph is simple, we denote this path by its vertex sequence $x_0, x_1, \ldots, x_n$ (because listing these vertices uniquely determines the path).
- The path is a *circuit* if it begins and ends at the same vertex, that is, if $u = v$, and has length greater than zero.
- The path or circuit is said to *pass through* the vertices $x_1, x_2, \ldots, x_{n-1}$ or *traverse* the edges $e_1, e_2, \ldots, e_n$.
- A path or circuit is *simple* if it does not contain the same edge more than once.
- Path in directed graphs is the same as in undirected graphs except that the path must go in the direction of the arrow.

Connectedness in Undirected Graphs

An undirected graph is called *connected* if there is a path between every pair of distinct vertices of the graph. An undirected graph that is not connected is called *disconnected*. We say that we *disconnect* a graph when we remove vertices or edges, or both, to produce a disconnected subgraph.



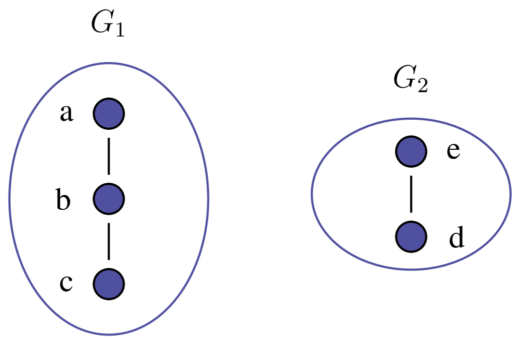
Connected Disconnected

Theorem 1

There is a simple path between every pair of distinct vertices of a connected undirected graph.

Connected Components

A connected component of a graph G is a connected subgraph of G that is not a proper subgraph of another connected subgraph of G. That is, a connected component of a graph G is a maximal connected subgraph of G. A graph G that is not connected has two or more connected components that are disjoint and have G as their union.



Cut Vertices

Cut vertices are vertices that produce a subgraph with more connected components when removed from a graph (and all incident edges to it). Removing a cut vertex v in in a connected graph G will make G disconnected.

Cut Edges/Bridges

Cut edges or *bridges* are edges that produce a subgraph with more connected components when removed from a graph. Removing a cut edge (u, v) in a connected graph G will make G disconnected.

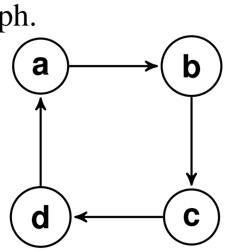
Connectedness in Directed Graphs

Strongly Connected

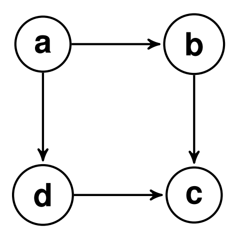
A *directed graph* is strongly connected if there is a path from a to b and from b to a whenever a and b are vertices in the graph.

Weakly Connected

A directed graph is *weakly connected* if there is a path between every two vertices in the underlying undirected graph.



Strongly connected



Weakly connected

Connected Components

The subgraphs of a directed graph G that are strongly connected but not contained in larger strongly connected subgraphs, that is, the maximal strongly connected subgraphs, are called the *strongly connected components* or *strong components* of G.

Paths and Isomorphism

The connectedness and the existence of a circuit or simple circuit of length k are graph invariants with respect to isomorphism. The existence of a simple circuit of a particular length can be used to show that two graphs are not isomorphic.

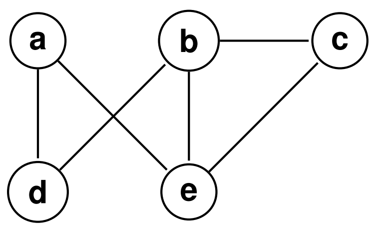
Counting Paths Between Vertices

Theorem 2

Let G be a graph with adjacency matrix A with respect to the ordering $v_1, v_2, \ldots, v_n$ of the vertices of the graph (with directed or undirected edges, with multiple edges and loops allowed). The number of different paths of length r from v_i to v_j , where r is a positive integer, equals the (i, j)th entry of A^r.

10.4 pg. 689 # 1

Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?



a) a, e, b, c, b

   Path of length 4. Not simple because edge $\{b, c\}$ is used twice. Not a circuit.

b) a, e, a, d, b, c, a

   Not a path, no edge between c and a

c) e, b, a, d, b, e

   Not a path, no edge between b and a

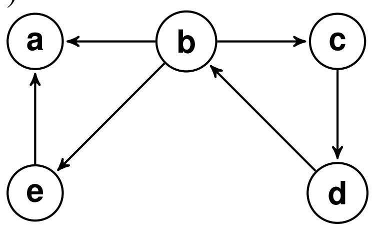
d) c, b, d, a, e, c

   Path of length 5. Is a simple path. Is a circuit.

10.4 pg. 689 # 11

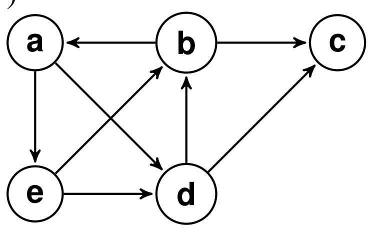
Determine whether each of these graphs is strongly connected and if not, whether it is weakly connected.

a)



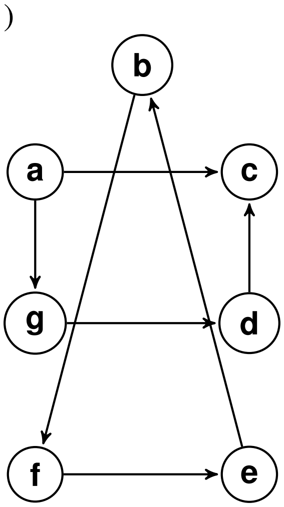
This graph is not strongly connected because there are no paths that start with a. This graph is weakly connected because the underlying undirected graph is connected.

b)



This graph is not strongly connected because there are no paths that start with c. This graph is weakly connected because the underlying undirected graph is connected.

c)

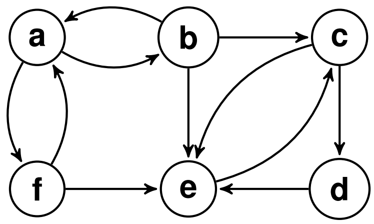


This graph is not strongly connected nor weakly connected because the underlying undirected graph is not connected.

10.4 pg. 690 # 15

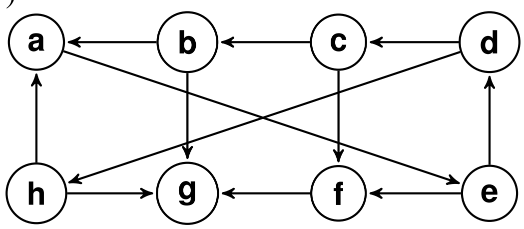
Find the strongly connected components of each of these graphs.

a)



$\{a, b, f\}$ and $\{c, d, e\}$ are the strongly connected components of the graph.

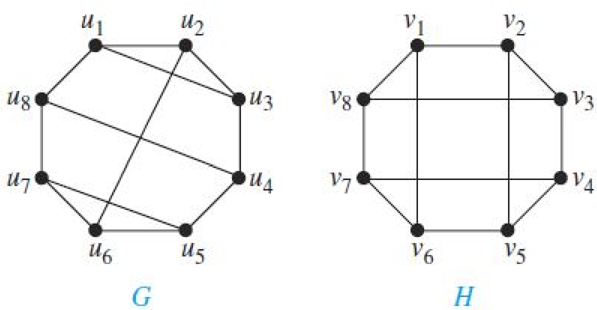
b)



$\{a, b, c, d, e, h\}$, $\{f\}$, and $\{g\}$ are the strongly connected components of the graph.

10.4 pg. 690 # 21

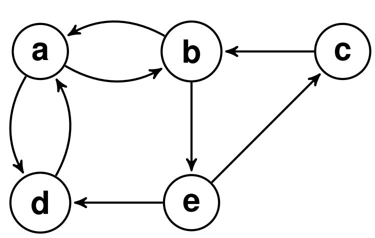
Use paths either to show that these graphs are not isomorphic or to find an isomorphism between them.



The graph G has a triangle formed by the path (u_1, u_2, u_3, u_1). In H, there is no such triangle formed by any of the vertices, therefore these graphs are not isomorphic.

10.4 pg. 691 # 27

Find the number of paths from a to e in the directed graph of length



a) 2.

Use Theorem 2. Compute A^2 and look at the $(1,5)$th entry.

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \; A^2 = \begin{bmatrix} 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

There is 1 path of length 2.

b) 3.

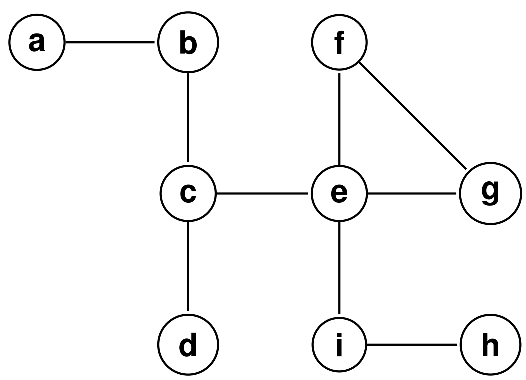
Same as question (a). Compute A^3 and look at the $(1,5)$th entry.

$$A^3 = \begin{bmatrix} 0 & 2 & 1 & 3 & 0 \\ 3 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

There are no paths of length 3.

10.4 pg. 691 # 33

Find all the cut vertices of the given graph.



The cut vertices are b, c, e, and i. Removing one of these vertices will disconnect the graph.