10.5 Euler and Hamilton Paths

Euler Circuit

An *Euler circuit* in a graph G is a simple circuit containing every edge of G.

Euler Path

An *Euler path* in G is a simple path containing every edge of G.

Theorem 1

A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has an even degree.

Theorem 2

A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

Algorithm for Constructing an Euler Circuit

Algorithm $euler(G$: connected multigraph with all vertices of even degree)

$circuit$ = a circuit in G beginning at an arbitrarily chosen vertex with edges successively added to form a path that returns to this vertex
$H = G$ with the edges of this circuit and all isolated vertices removed
while H has edges **do**
    $subcircuit =$ a circuit in H beginning at a vertex in G that also is an endpoint of an edge of $circuit$
    $H = H$ with edges of $subcircuit$ and all isolated vertices removed
    $circuit = circuit$ with $subcircuit$ inserted at the appropriate vertex
end while
return $circuit\{circuit$ is an Euler circuit$\}$

Hamilton Circuit

A *Hamilton circuit* is a simple circuit that traverses every vertex in G exactly once.

Hamilton Path

A *Hamilton path* is a simple path that traverses every vertex in G exactly once.

Dirac's Theorem

If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.

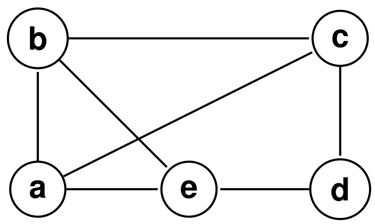
Ore's Theorem

If G is a simple graph with n vertices with $n \geq 3$ such that deg(u) + deg(v) $\geq n$ for every pair of nonadjacent vertices u and v in G, then G has a Hamilton circuit.

Note: Dirac's and Ore's theorem do not provide necessary conditions for the existence of a Hamilton circuit.

10.5 pg. 703 # 1

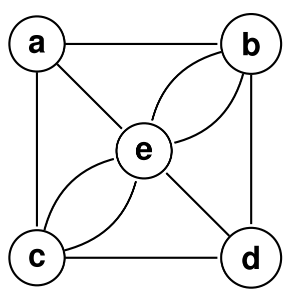
Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.



By theorem 1, we know this graph does not have an Euler circuit because we have four vertices of odd degree. By theorem 2, we know this graph does not have an Euler path because we have four vertices of odd degree.

10.5 pg. 703 # 3

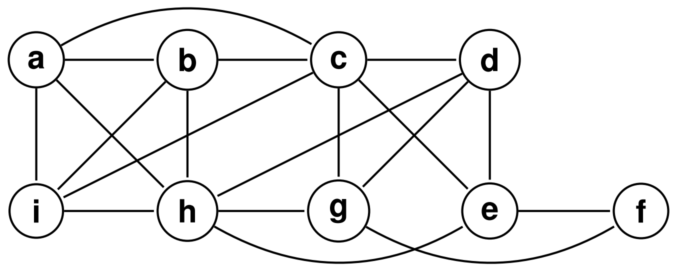
Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.



By theorem 1, this graph does not have an Euler circuit because we have two vertices with odd degrees (a and d). This graph does have an Euler path by Theorem 2. The path is as follows: $a, e, c, e, b, e, d, b, a, c, d$.

10.5 pg. 704 # 7

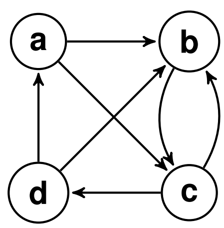
Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.



By theorem 1 there is an Euler circuit because every vertex has an even degree. The circuit is as follows: $a, b, c, d, e, f, g, h, i, a, h, b, i, c, e, h, d, g, c, a$

10.5 pg. 703 # 19

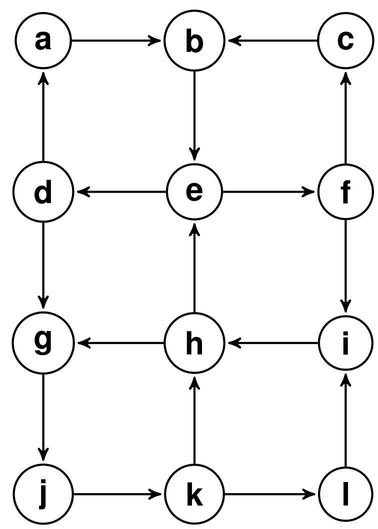
Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.



There is no Euler circuit because b has a different out-degree and in-degree. There is also no Euler path because b's in-degree and out-degree differ by more than 1.

10.5 pg. 703 # 23

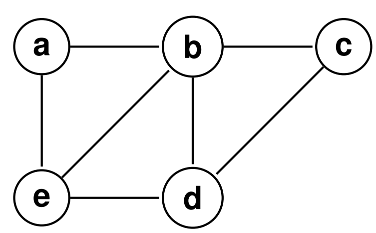
Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.



There is no Euler circuit because there are vertices that do not have the same in-degree and out-degree. We also do not have a Euler path because we have more than two vertices whose in-degree and out-degree that differ by 1.

10.5 pg. 705 # 31

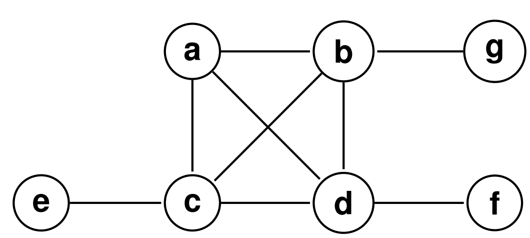
Determine whether the given graph has an Hamilton circuit. If it does, find such a circuit. It it does not, give an argument to show why no such circuit exists.



This graph has a Hamilton circuit. a, b, c, d, e, a is a circuit.

10.5 pg. 705 # 33

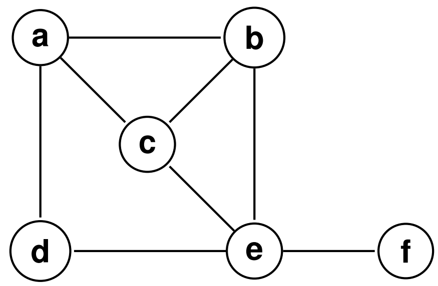
Determine whether the given graph has an Hamilton circuit. If it does, find such a circuit. It it does not, give an argument to show why no such circuit exists.



There is no Hamilton circuit because there are vertices of degree 1 (pendants) in the graph.

10.5 pg. 706 # 39

Determine whether the given graph has an Hamilton path. If it does, find such a path. It it does not, give an argument to show why no such path exists.

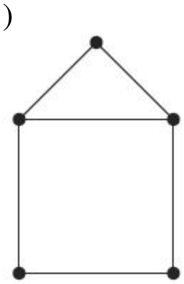


There is a Hamilton path, one such path is: f, e, d, a, b, c.

10.5 pg. 706 # 47

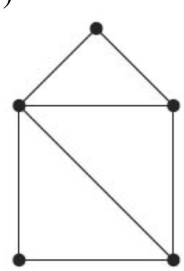
For each of these graphs, determine (i) whether Dirac's theorem can be used to show that the graph has a Hamilton circuit,(ii) whether Ore's theorem can be used to show that the graph has a Hamilton circuit, and (iii) whether the graph has a Hamilton circuit.

a)



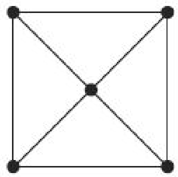
$n = 5$. We cannot apply Dirac's theorem because we have vertices with degree 2 and $2 < 5/2$. We also cannot apply Ore's theorem because there are two nonadjacent vertices with degree 2 and their sum is less than n, $2 + 2 < 5$. However, we can easily see that there is a Hamilton circuit by just traversing the vertices around the pentagon. This means that both theorems do not provide necessary conditions for the existence of a Hamilton circuit.

b)



Everything in part (a) can be applied to this problem.

c)



$n = 5$. By Dirac's theorem we have a Hamilton circuit because each vertex has a degree of 3 or 4, which is greater than $5/2$. Ore's theorem can also be applied because the smallest number for the degree of two nonadjacent vertices is 6, which is greater than or equal to 5.