9.6 Partial Orderings

A relation R on a set S is called a partial ordering or partial order if it is *reflexive*, *antisymmetric*, and *transitive*.

Poset

A set S together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R) or $(S, \preccurlyeq)$. Members of S are called *elements* of the poset.

Comparable

The elements a and b of a poset $(S, \preccurlyeq)$ are called *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$.

Incomparable

When a and b are elements of S such that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, a and b are called *incomparable*.

Totally Ordered and Total Order

If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a *totally ordered* or *linearly ordered set*, and $\preccurlyeq$ is called a *total order* or a *linear order*. A totally ordered set is also called a *chain*.

Well-Ordered Set

$(S, \preccurlyeq)$ is a *well-ordered set* if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

Lexicographic Ordering

Given two posets $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$, we construct a partial ordering on the Cartesian product of the two posets. The *lexicographic ordering* $\preccurlyeq$ on $A_1 \times A_2$ is defined by specifying that $(a_1, a_2) \prec (b_1, b_2)$ if and only if

- $a_1 \prec_1 b_1$ or
- $a_1 = b_1$ and $a_2 \prec_2 b_2$

Hasse Diagrams

A visual representation of a partial ordering.
To construct a Hasse diagram for a finite poset $(S, \preccurlyeq)$, do the following:

- Construct a digraph representation of the poset $(S, \preccurlyeq)$ so that all edges point up (except the loops)
- Eliminate all loops

- Eliminate all edges that are redundant because of transitivity
- Eliminate the arrows at the ends of edges since everything points up.

Minimal Elements

Let $(A, \preccurlyeq)$ be a poset. Then $a \in A$ is *minimal* in the poset if there is no element $b \in A$ such that $b \prec a$.

Maximal Elements

Let $(A, \preccurlyeq)$ be a poset. Then $a \in A$ is *maximal* in the poset if there is no element $b \in A$ such that $a \prec b$.

Note: There can be more than one minimal and and maximal element in a poset.

Least Element

Let $(A, \preccurlyeq)$ be a poset. Then $a \in A$ is the *least element* if for every element $b \in A$, $a \preccurlyeq b$.

Greatest Element

Let $(A, \preccurlyeq)$ be a poset. Then $a \in A$ is the *greatest element* if for every element $b \in A$, $b \preccurlyeq a$.

Upper Bound

Let $S \subseteq A$ in the poset $(A, \preccurlyeq)$. If there exists an element $u \in A$ such that $s \preccurlyeq u$ for all $s \in S$, then u is called an *upper bound* of S.

Lower Bound

Let $S \subseteq A$ in the poset $(A, \preccurlyeq)$. If there exists an element $l \in A$ such that $l \preccurlyeq s$ for all $s \in S$, then l is called a *lower bound* of S.

Least Upper Bound

If a is an upper bound of S such that $a \preccurlyeq u$ for all upper bound u of S then a is the *least upper bound* of S, denoted by $lub(S)$.

Greatest Lower Bound

If a is a lower bound of S such that $l \preccurlyeq a$ for all lower bound l of S then a is the *greatest lower bound* of S, denoted by $glb(S)$.

Lattices

A poset in which every pair of elements has both a least upper bound and a greatest lower bound is called a *lattice*.

Topological Sorting

A total ordering $\preccurlyeq$ is said to be *compatible* with the partial ordering R if $a \preccurlyeq b$ whenever aRb. Constructing a compatible total ordering from a partial ordering is called *topological sorting*. Use Lemma 1 for this.

Lemma 1: Every finite nonempty poset $(S, \preccurlyeq)$ has at least one minimal element.

Algorithm $topologicalSort((S, \preccurlyeq)$: finite poset)

$k = 1$
while $S \neq \emptyset$ **do**
  $a_k =$ minimal element of S
  $S = S - \{a_k\}$
  $k = k + 1$
end while
return $a_1, a_2, \ldots, a_n \{a_1, a_2, \ldots, a_n$ is the compatible total ordering of $S\}$

9.6 pg. 630 # 1

Which of these relations on $\{0, 1, 2, 3\}$ are partial orderings? Determine the properties of a partial ordering that the others lack.

a) $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$

This is a partial ordering.

b) $\{(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$

This is not a partial ordering. This relation is not antisymmetric because we have $(2, 3)$ and $(3, 2)$ in the relation.

c) $\{(0, 0), (1, 1), (1, 2), (2, 2), (3, 3)\}$

This is a partial ordering.

d) $\{(0, 0), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$

This is a partial ordering.

e) $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

This is not a partial ordering. This relation is not antisymmetric because we have $(0, 2)$ and $(2, 0)$ in the relation. This is relation is also not transitive because we are missing $(2, 1)$ for $(2, 0)$ and $(0, 1)$.

9.6 pg. 630 # 3

Is (S, R) a poset if S is the set of all people in the world and $(a, b) \in R$, where a and b are people, if

a) a is taller than b?

   This is not a poset because it is not reflexive. If we have a person a, then clearly a cannot be taller than himself/herself.

b) a is not taller than b?

   This is not a poset because it is not antisymmetric. Consider that we have a person a and a person b and $a \neq b$, then the order pairs (a, b) and (b, a) can exist in the relation because we can have a and b be the same height.

c) $a = b$ or a is an ancestor of b?

   This is a poset. This relation satisfies the reflexive property because of $a = b$. This relation also satisfies antisymmetric because if a is an ancestor of b, then it is obvious that b cannot be an ancestor of a. Lastly, this is transitive because if we have a is an ancestor of b and b is an ancestor of c, then clearly a is an ancestor of c.

d) a and b have a common friend?

   This is not a poset because it is not antisymmetric. Consider that you have two friends, a and b, then the ordered pairs (a, b) and (b, a) satisfies the relation.

9.6 pg. 630 # 5

Which of these are posets?

a) $(\mathbf{Z}, =)$

   This is a poset. The only ordered pairs we will have in this relation is (a, a) for all $a \in \mathbf{Z}$. This would mean that the relation is reflexive, antisymmetric, and transitive.

b) $(\mathbf{Z}, \neq)$

   This is not a poset because it is not reflexive. We cannot have the order pair (a, a) for all $a \in \mathbf{Z}$. This relation is also not antisymmetric and not transitive.

c) $(\mathbf{Z}, \geq)$

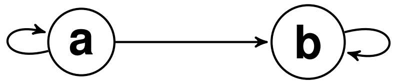
   This is a poset. For reflexive, we can have the ordered pair (a, a) for all $a \in \mathbf{Z}$. This is also antisymmetric because consider the ordered pair (a, b) and $a \neq b$, this would mean that $a > b$. If this is the case, then $b > a$ is not true and you cannot have (b, a). This is also transitive because if $a > b$, $b > c$, and $a \neq b \neq c$. Then it follows that $a > c$ for all $a, b, c \in \mathbf{Z}$.

d) $(\mathbf{Z}, \nmid)$

   This is not a poset because it is not reflexive. Consider $2 \nmid 2$, since this is not true, we cannot have $(2, 2)$. This relation is also not antisymmetric and not transitive.

9.6 pg. 630 # 11

Determine whether the relation with the directed graph shown is a partial order.



This is a partial order because it is reflexive, antisymmetric, and transitive.

9.6 pg. 630 # 19

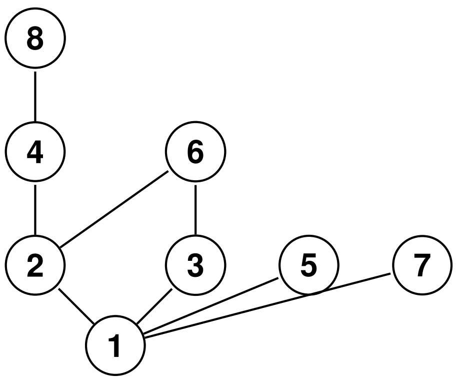
Find the lexicographic ordering of the bit strings $0, 01, 11, 001, 010, 011, 0001$, and 0101 based on the ordering $0 < 1$.

All the strings that begin with 0 precede all those that start with 1.
$0 < 0001 < 001 < 01 < 010 < 0101 < 011 < 11$

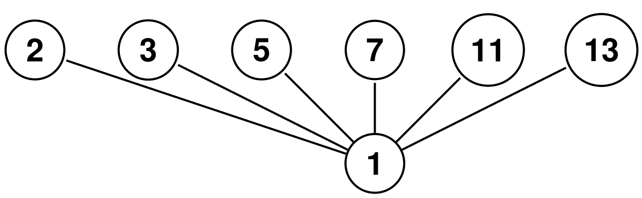
9.6 pg. 631 # 23

Draw the Hasse diagram for divisibility on the set

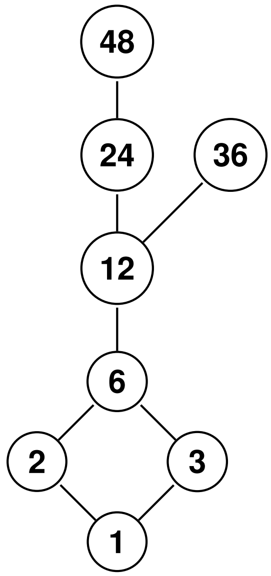
a) $\{1, 2, 3, 4, 5, 6, 7, 8\}$



b) $\{1, 2, 3, 5, 7, 11, 13\}$



c) $\{1, 2, 3, 6, 12, 24, 36, 48\}$

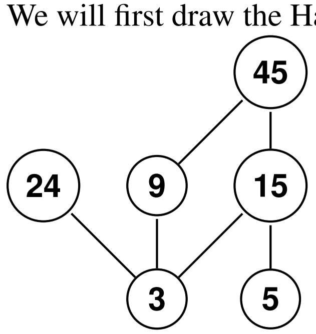


9.6 pg. 631 # 33

Answer these questions for the poset $(\{3, 5, 9, 15, 24, 45\}, |)$.

a) Find the maximal elements.

We will first draw the Hasse diagram.



Our maximal elements are 24 and 45.

b) Find the minimal elements.

Our minimal elements are 3 and 5.

c) Is there a greatest element?

There is no greatest element because this element would have to be a number that all other elements divide. Since our maximal elements are 24 and 45, and they do not divide each other, we do not have a greatest element.

d) Is there a least element?

There is no least element because this element would be a number that can divide all other elements. Since our minimal elements are 3 and 5, and they do not divide each other, we do not have a least element.

e) Find all upper bounds of $\{3, 5\}$.
   15 and 45.

f) Find the least upper bound of $\{3, 5\}$, if it exists.
   15.

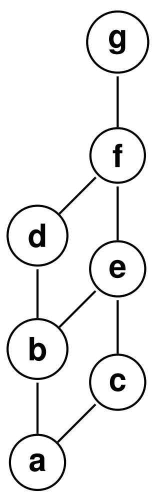
g) Find all lower bounds of $\{15, 45\}$.
   3, 5, and 15.

h) Find the greatest lower bound of $\{15, 45\}$, if it exists.
   15.

9.6 pg. 632 # 43

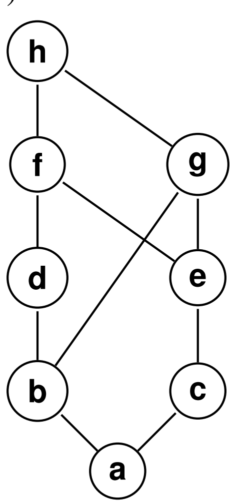
Determine whether the posets with these Hasse diagrams are lattices.

a)



Yes. Every two elements will have a least upper bound and greatest lower bound.

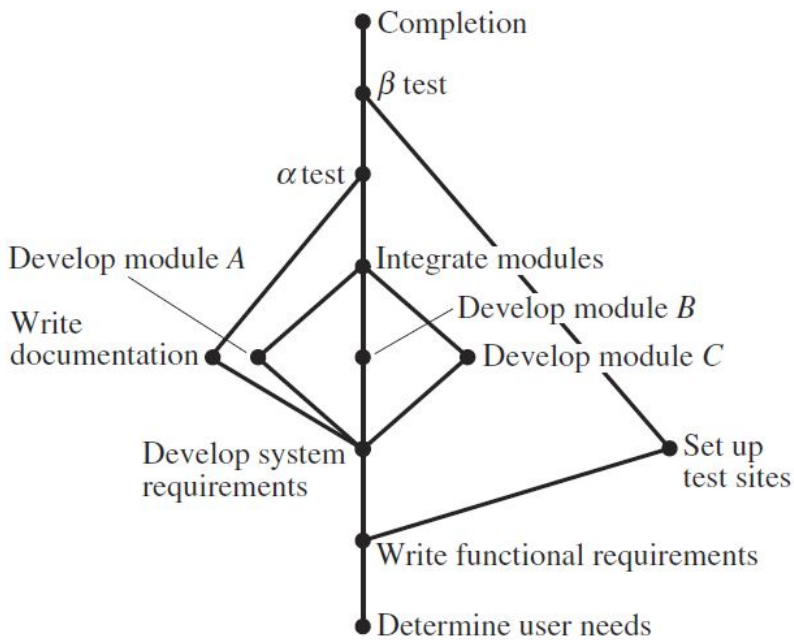
b)



No. If we take the elements b and c, then we will have $f, g,$ and h as the upper bound, but none of them will be the least upper bound.

9.6 pg. 633 # 67

Find an ordering of the tasks of a software project if the Hasse diagram for the tasks of the project is shown.



Simply work from the bottom to the top getting the minimal element each time (refer to topological sorting algorithm).

One such answer can be: Determine user needs $\prec$ Write functional requirements $\prec$ Set up test sites $\prec$ Develop system requirements $\prec$ Write documentation $\prec$ Develop module A $\prec$ Develop module B $\prec$ Develop module C $\prec$ Integrate modules $\prec$ α test $\prec$ β test $\prec$ Completion