9.1 Relations and Their Properties

Binary Relation

Definition: Let A, B be any sets. A *binary relation* R from A to B, written $R : A \times B$, is a subset of the set $A \times B$.

Complementary Relation

Definition: Let R be the binary relation from A to B. Then the complement of R can be defined by $\overline{R} = \{(a, b) | (a, b) \notin R\} = (A \times B) - R$

Inverse Relation

Definition: Let R be the binary relation from A to B. Then the inverse of R can be defined by $R^{-1} = \{(b, a) | (a, b) \in R\}$

Relations on a Set

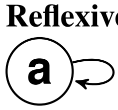
Definition: A *relation on a set* A is a relation from A to A. In other words, a relation on a set A is a subset of $A \times A$.

Digraph

Definition: A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a, b), and the vertex b is called the *terminal vertex* of this edge.

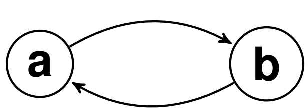
Properties

Reflexive: A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.



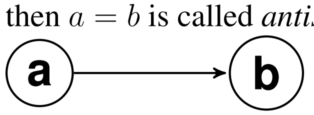
Every vertex has a self-loop.

Symmetric: A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.



If there is an edge from one vertex to another, there is an edge in the opposite direction.

Antisymmetric: A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.

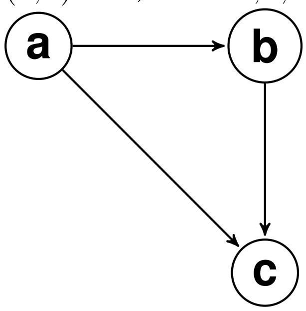


There is at most one edge between distinct vertices.

Some notes on Symmetric and Antisymmetric:

- A relation can be both symmetric and antisymmetric.
- A relation can be neither symmetric nor antisymmetric.

Transitive: A relation R on a set A is called *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.



If there is a path from one vertex to another, there is an edge from the vertex to another.

Combining Relations

Since relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined. Such as union, intersection, and set difference.

Composition

Definition: Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a, c), where $a \in A, c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

Powers of a Relation

Let R be a relation on the set A. The powers $R^n, n = 1, 2, 3, ...$, are defined recursively by $R^1 = R$ and $R^{n+1} = R^n \circ R$.

9.1 pg. 581 # 3

For each of these relations on the set $\{1, 2, 3, 4\}$, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.

a $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$

Not reflexive because we do not have $(1, 1)$, $(3, 3)$, and $(4, 4)$.
Not symmetric because while we we have $(3, 4)$, we do not have $(4, 3)$.
Not antisymmetric because we have both $(2, 3)$ and $(3, 2)$.
Transitive because if we have (a, b) in this relation, then a will be either 2 or 3. Then $(2, c)$

and $(3, c)$ are in the relation for all $c \neq 1$. Since whenever we have both (a, b) and (b, c), then we have (a, c) which makes this relation transitive.

b $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$

Reflexive because (a, a) is in the relation for all $a = 1, 2, 3, 4$.
Symmetric because for every (a, b), we have a (b, a).
Not antisymmetric because we have $(1, 2)$ and $(2, 1)$.
Transitive because while we have $(1, 2)$ and $(2, 1)$, we also have $(1, 1)$ and $(2, 2)$ in the relation.

c $\{(2, 4), (4, 2)\}$

Not reflexive because we do not have (a, a) for all $a = 1, 2, 3, 4$.
Symmetric because for every (a, b), we have a (b, a).
Not antisymmetric because we have both $(2, 4)$ and $(4, 2)$.
Not transitive because we are missing $(2, 2)$ and $(4, 4)$.

d $\{(1, 2), (2, 3), (3, 4)\}$

Not reflexive because we do not have (a, a) for all $a = 1, 2, 3, 4$.
Not symmetric because we do not have $(2, 1)$, $(3, 2)$, and $(4, 3)$.
Antisymmetric because for every (a, b), we do not have a (b, a).
Not transitive because we do not have $(1, 3)$ for $(1, 2)$ and $(2, 3)$.

e $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$

Reflexive because we have (a, a) for every $a = 1, 2, 3, 4$.
Symmetric because we do not have a case where (a, b) and $a \neq b$.
Antisymmetric because we do not have a case where (a, b) and $a \neq b$.
Transitive because we can satisfy (a, b) and (b, c) when $a = b = c$.

f $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$

Not reflexive because we do not have (a, a) for all $a = 1, 2, 3, 4$.
Not symmetric because the relation does not contain $(4, 1)$, $(3, 2)$, $(4, 2)$, and $(4, 3)$.
Not antisymmetric because we have $(1, 3)$ and $(3, 1)$.
Not transitive because we do not have $(2, 1)$ for $(2, 3)$ and $(3, 1)$.

9.1 pg. 581 # 7

Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if

a $x \neq y$.

Not reflexive because it's not the case $1 \neq 1$.
Is symmetric because $x \neq y$ and $y \neq x$.
Not antisymmetric because we have $x \neq y$ and $y \neq x$.
Not transitive because we can have $1 \neq 2$ and $2 \neq 1$ but not $1 \neq 1$.

b $xy \geq 1$.

Not reflexive because we can't have $(0, 0)$.
Is symmetric because we have $xy = yx$.
Not antisymmetric because we have $xy = yx$.
Is transitive because if we have $(a, b) \in R$ and that $(b, c) \in R$, it follows that $(a, c) \in R$. Note that in order for the relation to be true, a, b, and c will have to be all positive or all negative.

c $x = y + 1$ or $x = y - 1$.

Not reflexive because we can't have $(1, 1)$
Is symmetric because we have $x = y + 1$ and $y = x - 1$. They are equivalent equations.
Not antisymmetric because of the same reason above.
Not transitive because if we have $(1, 2)$ and $(2, 1)$ in the relation, $(1, 1)$ is not in relation.

g $x = y^2$.

Not reflexive because $(2, 2)$ does not satisfy.
Not symmetric because although we can have $(9, 3)$, we can't have $(3, 9)$.
Is antisymmetric because each integer will map to another integer but not in reverse (besides 0 and 1).
Not transitive because if we have $(16, 4)$ and $(4, 2)$, it's not the case that $16 = 2^2$.

h $x \geq y^2$.

Not reflexive because we can't have $(2, 2)$.
Not symmetric because if we have $(9, 3)$, we can't have $(3, 9)$.
Is antisymmetric, because each integer will map to another integer but not in reverse (besides 0 and 1).
Is transitive because if $x \geq y^2$ and $y \geq z^2$, then $x \geq z^2$.

9.1 pg. 582 # 27

Let R be the relation $R = \{(a, b) | a \mid b\}$ on the set of positive integers. Find

a R^{-1}

$R^{-1} = \{(b, a) | a \mid b\} = \{(a, b) | b \mid a\}$

b $\overline{R}$

$\overline{R} = \{(a, b) | a \nmid |b\}$.