9.4 Closure of Relations

Reflexive Closure

The *reflexive closure* of a relation R on A is obtained by adding (a, a) to R for each $a \in A$.

Symmetric Closure

The *symmetric closure* of R is obtained by adding (b, a) to R for each $(a, b) \in R$.

Transitive Closure

The *transitive closure* of R is obtained by repeatedly adding (a, c) to R for each $(a, b) \in R$ and $(b, c) \in R$.

Paths and Circuits in Directed Graphs

A *path* from a to b in the directed graph G is a sequence of edges $(x_0, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)$ in G, where n is a nonnegative integer, and $x_0 = a$ and $x_n = b$, that is, a sequence of edges where the terminal vertex of an edge is the same as the initial vertex in the next edge in the path. This path is denoted by $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ and has *length* n. We view the empty set of edges as a path of length zero from a to a. A path of length $n \geq 1$ that begins and ends at the same vertex is called a *circuit* or *cycle*.

Path in a Relation

Theorem 1: Let R be a relation on a set A. There is a path of length n, where n is a positive integer, from a to b if and only if $(a, b) \in R^n$.

Connectivity Relation A.K.A. Transitive Closures

Let R be a relation on a set A. The *connectivity relation* R^* consists of the pairs (a, b) such that there is a path of length at least one from a to b in R.
In other words:

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

where R^n consists of the pairs (a, b) such that there is a path of length n from a to b.

Theorem 2: The transitive closure of a relation R equals the connectivity relation R^*.

Theorem 3: Let M_R be the zero-one matrix of the relation R on a set with n elements. Then the zero-one matrix of the transitive closure R^* is

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \ldots \vee M_R^{[n]}$$

Simple Algorithm for Computing Transitive Closure

This algorithm shows how to compute the transitive closure. Runs in $O(n^4)$ bit operations.

Algorithm $transitive\_closure$(M_R : zero-one $n \times n$ matrix)

```
A = M_R
B = A
for i = 2 to n do
  A = A ⊙ M_R
  B = B ∨ A
end for
return B{B is the zero-one matrix for R*}
```

Warshall's Algorithm

Warhsall's algorithm is a faster way to compute transitive closure. Runs in $O(n^3)$ bit operations.

Algorithm $Warshall$(M_R : zero-one $n \times n$ matrix)

```
W = M_R
for k = 1 to n do
  for i = 1 to n do
    for j = 1 to n do
      w_ij = w_ij ∨ (w_ik ∧ w_kj)
    end for
  end for
end for
return W{W = [w_ij] is the zero-one matrix for R*}
```

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Let R be the relation on the set $\{0, 1, 2, 3\}$ containing the ordered pairs $(0, 1), (1, 1), (1, 2), (2, 0), (2, 2), (3, 0)$. Find the

a) reflexive closure of R

We need to add (a, a) in R to make a reflexive closure.
$\{(0, 0), (0, 1), (1, 1), (1, 2), (2, 0), (2, 2), (3, 0), (3, 3)\}$

b) symmetric closure of R

We need to add (b, a) for each (a, b) in R to make a symmetric closure.
$\{(0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2), (3, 0)\}$

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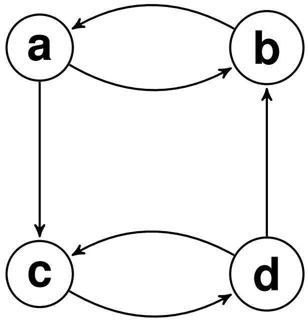
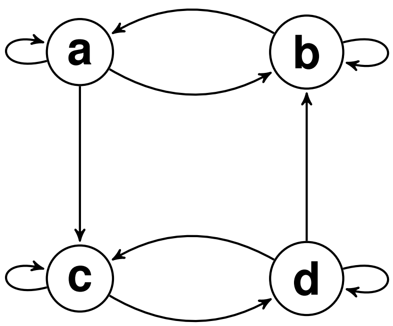
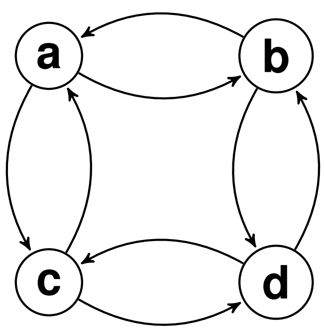
For the directed graph shown



a) Find the reflexive closure



b) Find the symmetric closure



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Use Algorithm 1 to find the transitive closure of these relations on $\{1, 2, 3, 4\}$.

a) $\{(1, 2), (2, 1), (2, 3), (3, 4), (4, 1)\}$

Transitive Closure

$=M_{R^*}$

$= M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee M_R^{[4]}$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

b) $\{(2,1),(2,3),(3,1),(3,4),(4,1),(4,3)\}$

Transitive Closure
$=M_{R^*}$
$= M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee M_R^{[4]}$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

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Use Warshall's algorithm to find the transitive closure of these relations on $\{1,2,3,4\}$.

a) $\{(1,2),(2,1),(2,3),(3,4),(4,1)\}$

$$W_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

b) $\{(2,1),(2,3),(3,1),(3,4),(4,1),(4,3)\}$

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad W_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad W_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad W_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$W_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$